GEOMETRY OF PURE STATES OF $N$ SPIN-$J$ SYSTEM
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ABSTRACT
We present the geometry of pure states of an ensemble of $N$ spin-$J$ systems using a generalisation of the Majorana representation. The approach is based on Schur-Weyl duality that allows for simple interpretation of the state transformation under the action of general linear and permutation groups.

MAJORANA REPRESENTATION
Majorana representation [Majorana(1932)] allows one to uniquely represent spin-$J$ state as $2J$ points on the Bloch-Poincare sphere.

The geometry is based on the stereographic projection:

- Arbitrary state $|\psi\rangle$ is connected with Bloch vector $\mathbf{a} = \langle \cos\theta \sin\psi, \sin\theta, \cos\theta \rangle$.
- The state of a spin-$J$ can be parametrised with a single complex number $z = e^{i\theta}\cos\theta/2$.

Resorting to stereographic projection each element of the set $|\psi\rangle$ can be drown on the Bloch-Poincare sphere. Examples are depicted in the representation (left) and multiplicity (right) spheres.

Figure 1: Stereographic projection
For a given state of spin-$J$:

$$|\psi\rangle = \sum_{m=-J}^{J} \psi_m |J,m\rangle$$

one can construct Majorana polynomial:

$$M(|\psi\rangle) = \sum_{m=-J}^{J} (-1)^m \binom{J}{m}^\dagger \psi_m^{+\dagger}$$

For each spin-$J$ state there exist a unique set of $2J$ complex numbers composed of $N$ roots of the Majorana polynomial $\{x_1, x_2, \ldots, x_N\}$ supplemented by $(2J-N)$-element set of $\infty$. Resorting to stereographic projection each element of the set can be drown on the Bloch-Poincare sphere. Examples are shown in figure 2.

(a) $|\psi\rangle = |6, -6\rangle + |6, 6\rangle$  (b) $|\psi\rangle = |3, -3\rangle + |3, 3\rangle$

Figure 2: Majorana representation of (a) the NOON state and (b) the octahedron state (optimal for the local reference frames alignment [Kolenderski(2010)])

Figure 3: (a) The geometry of an exemplary state $|\Psi\rangle$ depicted in the representation (left) and multiplicity (right) spheres. (b) Under the action of $U^{\text{ex}}$ only the representation sphere experiences a modification. (c) The logical qubit transformation, in general, changes the representation states.

SCHUR-WEYL DUALITY
Let us consider a general linear group element $g \in GL(2J+1, \mathbb{C})$, permutation group element $s \in S_N$ and its respective representations $\mathcal{M}(g), \tilde{\mathcal{S}}(s)$.

Theorem 1. The joint action of general linear and permutation groups $\tilde{\mathcal{M}}(g,s) = \mathcal{M}(g)\tilde{\mathcal{S}}(s)$ can be decomposed as:

$$\tilde{\mathcal{M}}(g,s) = \bigoplus_{\lambda \in \text{Part}(N,d)} \mathcal{M}(g)(\lambda) \otimes \tilde{\mathcal{S}}(s)(\lambda)$$

where $\mathcal{M}(g)\tilde{\mathcal{S}}(s)$ are irreducible representations (irreps) of $GL(2J+1, \mathbb{C})$ and $S_N$, respectively, and $\text{Part}(N,d)$ is a set of all partitions of $N$ into $d$ parts.

DECOMPOSITION

For simplicity, we consider here the case of $N$ spin-$\frac{1}{2}$ (N qubits) and a unitary evolution ($SU(2)$). The Hilbert space of such system can be decomposed as:

$$H^{N/2} = \bigoplus_{j=0}^{N/2} \mathcal{H}_j \otimes \mathcal{C}^j,$$

Moreover the action of unitary group is given by:

$$\tilde{\mathcal{U}}_j(\psi) \otimes |\psi\rangle = \bigoplus_{j=0}^{N/2} \sum_{\alpha \in A(2j)} c_{\alpha}^\dagger \tilde{\mathcal{U}}_j(\psi)|\alpha\rangle_j, |\alpha\rangle_j \otimes |\psi\rangle$$

where $\tilde{\mathcal{U}}_j(\psi)$ is an isrep of $g \in SU(2)$. Accordingly, an arbitrary state of N qubits can be represented as:

$$|\Psi\rangle = \bigoplus_{j=0}^{N/2} \sum_{\alpha \in A(2j)} c_{\alpha}^\dagger \tilde{\mathcal{U}}_j(\psi)|\alpha\rangle_j$$

It is seen that a state $|\Psi\rangle \in \{N\}^N$ is in one to one correspondence with the:

- representation state: $|\psi_{\alpha}\rangle \in \mathcal{H}_j$ and
- multiplicity state: $|\xi\rangle = \bigoplus_{\alpha \in A(2j)} c_{\alpha}^\dagger |\alpha\rangle_j$.

Each of representation states and the multiplicity state can be represented graphically using Majorana representation. This allows us to depict the state of $N$ qubits on:

- representation sphere, where all representation states $|\psi_{\alpha}\rangle$ are drawn together and
- multiplicity sphere, drawing the multiplicity state $|\xi\rangle$.

EXEMPLARY APPLICATION
We consider here an exemplary application in the theory of decoherence free subspaces. We assume that the logical qubit is encoded into the state $|\xi\rangle \in H^{N/2}$ of three physical qubits. In Majorana representation the action of the unitary rotation $U^{\text{ex}}$ can be seen as the rotation of all the points on the representation sphere as a solid body, whereas the points on the multiplicity sphere do not experience any modification.

Hence the logical qubit is entirely encoded in the multiplicity sphere and the „noisy evolution“ is reflected in the representation sphere.

For an exemplary state:

$$|\Psi\rangle = \frac{1}{2\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle) - (1 + \sqrt{3})|101\rangle + (1 - \sqrt{3})|010\rangle)$$

one can easily find the multiplicity state:

$$|\xi\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right)$$

and representation states:

$$|\psi_{001}\rangle_{1/2} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$
$$|\psi_{010}\rangle_{1/2} = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle).$$

A diagram summarising the presented discussion is depicted in figure 4. The exemplary state $|\psi\rangle$ and the action of the unitary $U^{\text{ex}}$ are depicted in panels (a) and (b), respectively. The last panel (c) shows the action of the unitary rotation of the logical qubit. $\tilde{\mathcal{U}}_1(\psi)|{001}\rangle = |001\rangle$, which in general affects the representation sphere.

The detailed discussion of the unitary and permutation group action can be found in Ref. [Kolenderski(2010)].

REFERENCES
[Majaron(1932)] E. Majorana, Nuovo Cimento 9, 43 (1932).
[Kolenderski(2010)] P. Kolenderski, OSID, in press.