PROJEKT DOKTORSKI, ACADEMIA SCIENTIARUM THORUNIENSIS

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Title of the project: Furstenberg systems of aperiodic multiplicative functions

Description: We consider the class $\mathcal{M} := \{\boldsymbol{u} : \mathbb{N} \to \mathbb{C} :\ \boldsymbol{u}$ is multiplicative and $|\boldsymbol{u}| \leq 1\}$ on which a "distance" (what is important is that it satisfies the triangle inequality) D is defined by setting

$$D(\boldsymbol{u}, \boldsymbol{v})^2 := \sum_{p \in \mathbb{P}} \frac{1}{p} \left(1 - \operatorname{Re}\left(\boldsymbol{u}(p)\overline{\boldsymbol{v}(p)}\right)\right).$$

When $D(\boldsymbol{u}, \boldsymbol{v}) < +\infty$ then one says that $\boldsymbol{u}$ *pretends to be* $\boldsymbol{v}$. *Non-pretentious* functions are those $\boldsymbol{u}$ which are at distance infinity from skewed Dirichlet characters, i.e. $D(\boldsymbol{u}, n^{it} \cdot \chi) = +\infty$ for $t \in \mathbb{R}$ and each Dirichlet character χ. The Möbius and Liouville functions $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$, respectively, are non-pretentious but $\boldsymbol{\mu}^2$ (the characteristic function of the set of square-free numbers) pretends to be $\mathbf{1}$. A multiplicative function $\boldsymbol{u} : \mathbb{N} \to \mathbb{C}$ is called *aperiodic* if $\lim_{N\to\infty} \frac{1}{N} \sum_{n \leq N} \boldsymbol{u}(an + b) = 0$ for each $a, b \geq 1$. By Delange theorem, each non-pretentious function is aperiodic, in particular, classical multiplicative functions as $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are aperiodic. In the 1960th Chowla and then Elliot stated famous conjectures about multiple auto-correlations of aperiodic functions capturing a common believe of their random nature. It is however only Sarnak in 2010 who phrased precisely that such purely (analytic) number theory problems are equivalent to some purely ergodic theory problems. Given any $\boldsymbol{u} : \mathbb{Z} \to \mathbb{D}$ ($\mathbb{D}$ stands for the unit disk, and $\boldsymbol{u}(-n) = \boldsymbol{u}(n)$) one can naturally define the subshift $(X_{\boldsymbol{u}}, S)$, where S denotes the left shift on $\mathbb{D}^{\mathbb{Z}}$ and $X_{\boldsymbol{u}}$ stands for the closure of the orbit of $\boldsymbol{u}$ under S. Now, $\boldsymbol{u} \in X_{\boldsymbol{u}}$ determines so called *empiric measures*: $(1/N) \sum_{n \leq N} \delta_{S^n \boldsymbol{u}}$ on $X_{\boldsymbol{u}}$, and using compactness of the space of measures, we can consider the set $V(\boldsymbol{u})$ of all limit points of empiric measures. Now, each $\kappa \in V(\boldsymbol{u})$ is S-invariant and the measure-theoretic system $(X_{\boldsymbol{u}}, \kappa, S)$ is called a *Furstenberg system of* $\boldsymbol{u}$. Returning to $\mathcal{M}$, Sarnak observed that the Chowla (Elliot) conjecture is equivalent to certain strong statistical properties of Furstenberg systems of $\boldsymbol{u}$ (relative Bernoulli property over the factor determined by $|\boldsymbol{u}|$). Recently Matomäki, Radziwiłł and Tao constructed an aperiodic $\boldsymbol{u} \in \mathcal{M}$ which **does not** satisfy Chowla conjecture, however they still conjecture that when aperiodicity is replaced by so called strong aperiodicity then we should expect the validity of both conjectures. Since then, in the strong aperiodicity case, an impressive progress has been made toward a possible proof.

The main idea of the thesis is however to stay on the level of aperiodic but not strongly aperiodic functions from $\mathcal{M}$ to determine to which extent the Chowla (Elliot) conjecture "survives" on the level of Furstenberg systems. The main subject is to study ergodic properties of the corresponding Furstenberg systems. Main directions of research (they are not independent of each other) are the following:

- Show that for each aperiodic $\boldsymbol{u} \in \mathcal{M}$ in the class of locally Archimedean functions of Matomäki, Radziwiłł and Tao there is a Furstenberg system which does not have discrete spectrum.
- Study entropy in this class. If it is positive, is there a Furstenberg system of $\boldsymbol{u}$ which is Bernoulli?
- Is there an aperiodic $\boldsymbol{u} \in \mathcal{M}$ which is completely deterministic (i.e. all of its Furstenberg systems have measure-theoretic entropy zero)?
- Are there any relations between Furstenberg systems of $\boldsymbol{u}$ and of $\boldsymbol{v}$ which pretends to be $\boldsymbol{u}$?
- Do strong topological assumptions on topological properties of $(X_{\boldsymbol{u}}, S)$ (e.g. minimality, unique ergodicity) imply strong restrictions on $\boldsymbol{u} \in \mathcal{M}$?