Spectral properties of entanglement witnesses and separable states

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Abstract. Entanglement witnesses are observables which when measured, detect entanglement in a composed system. It is shown what kind of relations among eigenvectors an observable should fulfill to be an entanglement witness. A similar analysis shows some relations among eigenvalues and eigenvectors of a density operator, which are necessary for the operator to be separable.

1. Introduction

The most important information resource in quantum-information science, which distiguishes it from classical information theory, is entanglement. It is important to check whether a given state is entangled, i.e. whether it can be used in a given quantum-information protocol. Despite the fact that there are different incomparable types of entanglement, at present we cannot even judge whether a given mixed state is separable or not. One of the ways of detecting entanglement is provided by a class of special observables called *entanglement witnesses*. This paper discusses the question of how separability of a given state determines its spectral properties and what spectral properties entanglement witnesses have.

From now we will consider only finite-level quantum systems. Having a quantum system which is composed of two subsystems, the Hilbert space of the system is a tensor product of the subsystems $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. In fixed bases of subsystems, any element of the space of the tensor product can be written as a matrix of its coordinates. For a given vector Ψ, we will denote this matrix by $\mathfrak{A}(\Psi)$. We define a *Schmidt rank* of a vector in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ as the rank of its coordinate matrix. The set of vectors of Schmidt rank 1 will be denoted by $\mathcal{S}_1$. Such vectors can be written as $\phi \otimes \psi$ for some vectors in the Hilbert spaces of the subsystems.

We define *pure separable states* as projectors onto vectors in $\mathcal{S}_1$. Following Werner [1], we extend this definition to mixed states, defining the set of mixed separable states as a convex hull of the set of separable pure states. Namely a state is separable when it can be written as a convex combination of projectors onto vectors of Schmidt rank 1: $\rho = \sum_i \lambda_i |\phi_i \otimes \psi_i\rangle\langle\phi_i \otimes \psi_i|$. States which are not separable are called entangled.

The problem of separability is then the problem of membership in a convex set whose extremal points are given. One of the ways to handle this problem is a concept of *entanglement witnesses* introduced in [2]. We define the dual cone of the subspace of separable states: $\{W : \ \forall\rho \in \mathcal{S}_1 \ \langle\rho|W\rangle_{HS} \geq 0\}$. The members of this cone, which are not positive, are called entanglement witnesses. The state is separable iff the mean value of all entanglement witnesses in this state is positive. For a given entangled state, there exists an entanglement witness whose mean value on the given state is negative. The kernel of this state is a hypersurface, which separates two convex sets - the set of separable states and the singleton of the given state.

The observable W fulfills then two obvious conditions:

- $\forall \Psi \in \mathcal{S}_1 \ \langle \Psi | W | \Psi \rangle \geq 0$
- $\langle \rho | W \rangle_{HS} < 0$

An observable which fulfills these two conditions is called *entanglement witness.* A state ρ which fulfills the condition $\langle \rho | W \rangle_{HS} < 0$ is said to be detected by W.

The problem of classification of entanglement witnesses remains unsolved, in general. However, in low dimensions (2×2, 2×3) we have such a classification. Any entanglement witness is of the form: $W = A^\Gamma + B, \quad A, B \geq 0$, where Γ denotes partial transpose in one of the subsystems (see [3]). Such witnesses are called *decomposable*, and states which can be detected by witnesses from this class are called NPT entangled states. In higher dimensions this class of entanglement witnesses is a proper subset of the set of all witnesses. We have then entangled states not detected by this class. They are called PPT entangled states. A simple criterion exists to check whether a given state is NPT (see [4], [5]). The most interesting are then non-decomposable witnesses and tools to detect PPT entanglement based on them.

2. Spectral properties of entanglement witnesses

Having a given hermitian observable W, one can decompose its domain according to its spectral decomposition: $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} = V_+ \oplus V_- \oplus V_0$, where the *positive subspace* V_+ is spaned by eigenvectors of positive eigenvalues, the *negative subspace* V_- is spaned by eigenvectors of negative eigenvalues, and V_0 is the kernel of W. In similar way, one can write W as a difference of two positive operators: $W = W_+ - W_-$, where W_+ is W restricted to V_+, and W_- is W restricted to V_-.

For further considerations yet another notation will be very usefull. For a given state Ψ, we define $\tilde{V}_\Psi = \mathrm{ImTr}_2 |\Psi\rangle\langle\Psi| \otimes \mathbb{C}^{d_2} \oplus \mathbb{C}^{d_1} \otimes \mathrm{ImTr}_1 |\Psi\rangle\langle\Psi|$.

Necessary condition: When an observable W is an entanglement witness, then the following conditions should be fulfilled:

(i) $V_- \supsetneq \{0\}$
(*W* should be able to detect anything)

(ii) $\forall \Psi \in \mathcal{S}_1 \ \Psi \in V_0 \oplus V_- \Rightarrow \Psi \in V_0$
(the mean value of W in any separable state cannot be negative)

(iii) $\forall \Psi \in \mathcal{S}_1 \cap V_0 \ \tilde{V}_\Psi \cap V_- \oplus V_0 \subset V_0$

Making the third condition stronger: $\forall \Psi \in \mathcal{S}_1 \cap V_0 \ V_- \in \tilde{V}_\Psi^\perp$, one gets a kind of weak inversion of the above theorem:

Sufficient condition: When the conditions (i) - (iii) are fulfilled, the third one in its stronger form, then for λ large enough the observable $\lambda W_+ - W_-$ is an entanglement witness.

Conditions (i) - (iii) with the intersection theorem for varieties in complex projective space (see [6]) give some restrictions on the signatures of entanglement witnesses.

Similarly, one can define the set of $k-$separable states as the convex hull of projectors onto Schmidt rank k vectors (see [7]), and the corresponding set of $k-$witnesses (see [8]). Both the necessary and sufficient conditions, as well as algebraic-geometric constraints mentioned above have straighforward generalization for $k-$witnesses.

3. Parametrization of the boundary of the set of entanglement witnesses
The set of operators Y which take positive mean value on vectors of Shmidt rank 1 can be divided into the convex subset of positive operators and its complement - the set of entanglement witnesses. To detect entanglement, it is enough to use witnesses from the boundary of Y- any other entanglement witness is a combination of the identity and some witness from the boundary, so set of states detected by such a witness is a proper subset of set of states detected by the witnesses from the boundary.

One can assign to any density operator ρ the entanglement witness from the boundary of Y of the form:

$$W_\rho = ||\sqrt{\rho}||^2_{S-sup} I - \rho \tag{1}$$

The operator ρ is positive, so its square root is defined properly. The norm is defined as follows: $||X||_{S-sup} = \sup\{X\Psi : ||\Psi|| = 1 \ \wedge \ \Psi \in \mathcal{S}_1\}$, and the following inequality takes place: $||X||_{S-sup} \leq ||X||_{sup}$.

In the case when $||\sqrt{\rho}||^2_{S-sup} = ||\sqrt{\rho}||^2_{sup}$, the boundary of the set of witnesses meets the boundary of the set of states.

4. Spectral properties of separable states
State ρ is separable iff the mean values of all entanglement witnesses on it are positive. As it was shown in the previous section, one can restrict the quantified set to the boundary of the set of observables which are positive on vectors from $\mathcal{S}_1$, parameterized by (1). Using this parametrizations, one gets the equivalent definition of separability of ρ:

$$\forall \eta \ \ \langle \eta | \rho \rangle_{HS} \leq ||\sqrt{\eta}||^2_{S-sup} \tag{2}$$

Restricting now the quantified set to some special classes of density operators, one can get some neccesary conditions for separability.

First class which is worth to distiguish is the class of normalized projectors, i.e. density operators which have all their non-zero eigenvalues degenerated. For a projector onto subspace V we have the formula: $||\sqrt{\Pi_V}||^2_{S-sup} = ||\Pi_V||^2_{S-sup} = \sup\{||\mathfrak{A}(\Psi)||_{sup} : \Psi \in V \ \wedge \ ||\Psi|| = 1\}$.

In the special case of one-dimensional subspace V, spaned by a vector Ψ, the above formula reduces to $||\sqrt{\Pi_V}||^2_{S-sup} = ||\mathfrak{A}(\Psi)||_{sup}$, which is equal to the maximal Schmidt coefficient of the vector Ψ. Then one gets the separability criterion:

$$\forall \Psi \ \ \langle \Psi | \rho | \Psi \rangle \leq ||\mathfrak{A}(\Psi)||^2_{sup}. \tag{3}$$

This criterion detects all separable pure states but is to weak to detect any PPT entangled state. Nevertheless it leads to some results having an interesting physical interpretation.

Properties of statistical mixtures: Consider a mixing machine which, using the system of beam splitters, gives as the output the mixture of states from n inputs. States from any input can occur in the output with equal probabilities. On the input i there is a source of pairs of particles in some pure entangled state $|\Psi_i\rangle\langle\Psi_i|$. We can assign to the sources their normalized efficiencies p_i. The output state of the machine is the density operator $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$. Now we want to find some restrictions on the efficiencies p_i as a neccesary condition for the separability of ρ.

Let us apply the condition (3) to the state $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, and then restrict the inequality to one element of the sum:

$$p_i |\langle \phi | \Psi_i \rangle|^2 \leq \langle \phi | \sum_i p_i (|\Psi_i\rangle\langle\Psi_i|) | \phi \rangle \leq ||\mathfrak{A}(\phi)||^2_{sup}$$

In particular, it should be fullfiled for $\phi = \Psi_i$, and it leads to the condition $\lambda_i \leq \mathfrak{A}(\Psi_i)$.

Properties of the spectrum of a separable state: A special case of statistical mixture is the spectral decomposition. Using the previous result to spectral decomposition one gets a criterion for the eigenvalues:

When a state is separable, then its eigenvalue is not greater than the supremum norm of the coordinate matrix of the corresponding eigenvector.

Example: Consider the density operators acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$, whose eigenbasis is the Bell (magic) basis:

$$\mathfrak{A}(\Phi_1) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathfrak{A}(\Phi_2) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathfrak{A}(\Phi_3) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathfrak{A}(\Phi_4) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

It is very easy to calculate the supremum norms of the matrices of coordinates of the basis vectors. Using the criterion for eigenvalues one gets: $\lambda_i \leq 1/2$. For this eigenbasis, it is also sufficient condition (see [9], [10]).

Generalization using higher-dimensional projectors: Having a spectral decomposition $\rho = \sum_{i \in I_0} p_i |\Psi_i\rangle\langle\Psi_i|$, one can define for $I \subset I_0$ a subspace $V_i = \mathrm{span}\{\Psi_\mathrm{i} : \ \mathrm{i} \in \mathrm{I}\}$. Using mutual orthogonality of eigenvectors, one gets the condition: $\sum_{i \in I} \lambda_i \leq ||\Pi_{V_I}||^2_{S-sup}$.

If the power of the set I is large enough, then the intersection theorem mentioned before implies the existence of a separable vector in V_I. Then the right-hand side is equal to one, and the condition becomes trivial. It is worth noticing that this criterion is already able to detect some PPT entangled states - it is based on entaglement witnesses with all negative and positive eigenvalues degenerated. According to a paper by B. M. Terhal [11], one can construct indecomposable witness with such a property.

5. Conclusions

It was shown that when an observable is an entanglement witness, it fulfills some necessary conditions concerning its kernel and its negative subspace. It was also shown that when one of the conditions is strengthened, then it guarantees that the given observable is an entanglement witness, possibly after rescaling its positive part. Although among isospectral density martices are both separable and entangled states, one can provide some necessary conditions for a density matrix with given spectrum to be separable, which uses the propetries of its invariant subspaces. A hierarchy of conditions is presented, where every weakening provides a condition which is simpler to check. Morover, such a conditions can be applied not only to the spectral decomposition, but to any statistical mixture, possibly consisting of states of non-orhogonal supports.

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