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Geometry of Local Orbits in Three-Qubit Problem

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Abstract. The geometry of orbits of a group of local operations is analyzed. It depends on the form of orbit stabiliser and is found by knowing the stabiliser. Discrete stabilisers for generic orbits are obtained. We prove that local group acts freely on a generic orbit.

1. Introduction

1.1. Physical situation

Consider three two-level systems (qubits), initially interacting, and then separated. Now the dynamics of the system is given by a one-parameter, strongly continuous subgroup of the group of local operations $U(2) \times U(2) \times U(2)$. The local group does not act transitively on $\mathbb{C}P^7$, so it splits the space of states of the system into local orbits. The orbit cannot be changed by a local operation, and becomes fixed, when the interaction between subsystems disappears. Local orbit is then the domain of any prescribed dynamics. The dynamics of entangled qubits gives us knowledge about properties of correlations among them. Topological properties of the domain of the dynamics are closely related with properties of the dynamics itself. Topological properties of an orbit tell us a lot about general local dynamics on this orbit.

1.2. Mathematical tools used

In general, the local group does not act freely, i.e. there are non-identity operations which do not move points in an orbit. Such operations for a fixed orbit form a subgroup which is called the stabiliser of this orbit. All non-discrete stabilisers were found in [1]. In this paper, all discrete stabilisers are obtained (stabilisers of orbits of full dimension). The geometry of all orbits will be found using only the form of stabiliser, by aliasing points differing by an element of the stabiliser.

The orbits have topological structure of fibre bundles. A bundle which belongs to this class has a typical fibre. Let us recall that a fibre bundle is a generalization of cartesian product. In cartesian product there are two projections to both factors. In total space of a bundle there is only one projection to the base of the bundle, but in general there's no possibility to define the second projection to the fibre. When it is possible, the bundle is homeomorphic to a cartesian product of the base and the fibre and is called a trivial bundle. When it is not possible, the bundle is called nontrivial. The simplest example of nontrivial bundle is the universal covering of S^1, given by the exponential function $\phi : \mathbb{R} \to S^1$, $\phi(x) = e^{ix}$. The fibre over a point, i.e. the preimage of the projection, is $\mathbb{Z}$, and the base is S^1. The covering space $\mathbb{R}$ is connected but $S^1 \times \mathbb{Z}$ — the family of circles parameterized by $\mathbb{Z}$ — is not connected. It means that this bundle cannot be homeomorphic to a cartesian product of the base and the fibre. The bundle will be denoted by the base $\coprod$ the fibre.

The fundamentals of fibre bundle theory can be found in [2]. The basic facts of algebraic topology can be found in e.g. [3].

1.3. Review of existing results

The geometry of all local orbits of two entangled systems of dimension lower than 4 was found in [4] by using the non-uniqueness of Schmidt decomposition. This method cannot be extended to three entangled qubits, because Schmidt decomposition does not work for more than two particles (see [5]).

The problem of geometry of orbits of two entangled qubits was considered in [6] using the second Hopf bundle. The orbits of separable and maximally entangled states was found there. The geometry of an orbit of an entangled (but not maximally entangled) state can be found by a simple adaptation of the method used for an orbit of the maximally entangled state.

The analogous structure describing three qubits is the third Hopf bundle, considered in [7]. That work, however, does not touch the problem of the geometry of orbits.

2. Manifold of Local Operations on the Set of States

Consider an element of the group U(2). It can be written as $e^{i\phi} \times U$, where $U \in \mathrm{SU}(2)$ and $\phi \in [0, 2\pi)$. The symbol $\times$ denotes here and from now on direct product of elements of the respective groups. This representation is not unique. To get a unique representation, one should alias antipodal points of such manifold. It gives $S^1 \times S^3/\mathbb{Z}_2 = S^1 \coprod \mathbb{R}P^3$.

This bundle is not trivial. Suppose that on the contrary $S^1 \times S^3/\mathbb{Z}_2 = S^1 \times \mathbb{R}P^3$. One of the known facts from algebraic topology says, that

$$\Pi(A \times B) \;=\; \Pi(A) \times \Pi(B)\,. \tag{1}$$

It implies that the projection of a loop in U(2) should be a loop in the fibre. Consider a path in $S^1 \times S^3$ — the universal covering of U(2) joining points $(e^{i\phi}, U)$ and $(e^{i(\phi+\pi)}, -U)$. These points represent the same element of U(2), so the considered path is a loop in total space. The fundamental group of the base of this bundle — $\mathbb{R}P^3$ is $\mathbb{Z}_2$. The projection of the loop represents 1 in the fundamental group of the base space. When going along this loop, a point in the fibre changes by π, so the projection of the loop in the fibre is not closed. For this loop formula (1) does not work, so the contradiction implies, that the bundle is not homeomorphic with a cartesian product of the base and the fibre, so is not trivial.

The symbol $\coprod$ means that U(2) is a bundle with base space $\mathbb{R}P^3$ and the fibre S^1.

Consider now an element of U(2) × U(2). It can be written as $e^{i\phi} \times U \times V$, where $U, V \in$ SU(2). Again, to get unique representation one should alias antipodal points on both manifolds. It gives $S^1 \times S^3/\mathbb{Z}_2 \times S^3/\mathbb{Z}_2 = S^1 \coprod (\mathbb{R}P^3 \times \mathbb{R}P^3)$. This bundle is not trivial again. Going along the loops representing the elements $(1, 0)$ and $(0, 1)$ of the fundamental group, the parametrization of the fibre changes by π.

Similarly the manifold of the U(2) × U(2) × U(2) group is $S^1 \coprod (\mathbb{R}P^3)^3$. The parametrization of the fibre changes, when going along loops representing the elements

$$(1,0,0), (0,1,0), (0,0,1), (1,1,1) \;\in\; \Pi(\mathbb{R}P^3 \times \mathbb{R}P^3 \times \mathbb{R}P^3) = (\mathbb{Z}_2)^3\,.$$

By aliasing the vectors representing the same state (differing by a phase factor), each fibre collapses to a point, and the manifold of the group of local operations on the set of states of one, two, or three qubits is a proper cartesian power of $\mathbb{R}P^3$.

3. The Geometry of Orbits in Two Qubit Problem

It is possible now to find the geometry of orbits of local operations on the set of states of two qubits using only the knowledge about the form of a stabiliser of such orbits. In case of two qubits, the set of orbits is parameterized by one real parameter — the concurrence (see [8]). There are only three different types of orbits: when the concurrence is 0 (separable states), when the concurrence is 1 (maximally entangled states) and for remaining values of the concurrence (entangled, but not maximally entangled states). Each of that types has different stabiliser. Admissible forms of stabilisers were found in [1].

3.1. Separable states

The stabiliser of this orbit has the form $e^{i\theta\sigma_z} \times e^{i\phi\sigma_z}$, where θ and ϕ are arbitrary angles, so the elements of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ (the universal covering of $\mathbb{R}P^3 \times \mathbb{R}P^3$) of the form [1]

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \times \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \cdot \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix}$$
$$= \begin{bmatrix} a \cdot e^{i\theta} & b \cdot e^{i\theta} \\ -\bar{b} \cdot e^{-i\theta} & \bar{a} \cdot e^{-i\theta} \end{bmatrix} \times \begin{bmatrix} c \cdot e^{i\phi} & d \cdot e^{i\phi} \\ -\bar{d} \cdot e^{-i\phi} & \bar{c} \cdot e^{-i\phi} \end{bmatrix}$$

[1] The "$\times$" denotes here the direct product of group elements represented by above matrixes. If however the reader is inclined to think in terms of representations, $\times$ thould be replaced by $\otimes$.

represent the same state for all θ and ϕ. Each element of such torus has its antipodal element on the same torus, so aliasing the points on the torus aliases the antipodal points in $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

As a representant of a point of the orbit, the element where $\arg a = -\theta$ and $\arg c = -\phi$ can be taken. After redefining the parameters:

$$\begin{array}{ccc} a \cdot e^{i\theta} & \rightarrow & a \\ b \cdot e^{i\theta} & \rightarrow & b \\ c \cdot e^{i\phi} & \rightarrow & c \\ d \cdot e^{i\phi} & \rightarrow & d \end{array} \tag{2}$$

such representant can be written as:

$$\begin{bmatrix} a & b \\ -\bar{b} & a \end{bmatrix} \times \begin{bmatrix} c & d \\ -\bar{d} & c \end{bmatrix}, \tag{3}$$

where $a, c \in \mathbb{R}_+$ and $b, d \in \mathbb{C}$. A pair (a, b) represents a point from a closed two-dimensional hemisphere, similarly for the pair (c, d). When $a = 0$, the first matrix in the product (3) has the form

$$\begin{bmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{bmatrix}.$$

All such matrices differ by an element of the stabiliser, so they represent the same point of the orbit, and similarly for the second qubit. Aliasing the points from the boundary of the hemisphere gives the sphere, so the set of orbits in this case is $S^2 \times S^2$ (the product of Bloch spheres of both qubits).

3.2. Entangled, but not maximally entangled states

The stabiliser of this orbit has the form $e^{i\nu\sigma_z} \times e^{-i\nu\sigma_z}$. Any element of SU(2)×SU(2) of the form

$$\begin{bmatrix} e^{i\nu} & 0 \\ 0 & e^{-i\nu} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \times \begin{bmatrix} e^{-i\nu} & 0 \\ 0 & e^{i\nu} \end{bmatrix} \cdot \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix}$$
$$= \begin{bmatrix} a \cdot e^{i\nu} & b \cdot e^{i\nu} \\ -\bar{b} \cdot e^{-i\nu} & \bar{a} \cdot e^{-i\nu} \end{bmatrix} \times \begin{bmatrix} c \cdot e^{-i\nu} & d \cdot e^{-i\nu} \\ -\bar{d} \cdot e^{i\nu} & \bar{c} \cdot e^{i\nu} \end{bmatrix}$$

represents the same state for all ν. As a representant of a point of the orbit, the element with $\arg a = -\nu$ can be taken. After redefining the parameters:

$$\begin{array}{rcl} a \cdot e^{i\nu} & \rightarrow & a \\ b \cdot e^{i\nu} & \rightarrow & b \\ c \cdot e^{-i\nu} & \rightarrow & c \\ d \cdot e^{-i\nu} & \rightarrow & d \end{array} \tag{4}$$

such representant can be written as

$$\begin{bmatrix} a & b \\ -\bar{b} & a \end{bmatrix} \times \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix}. \tag{5}$$

The set of pairs (a, b) forms again the Bloch sphere (after aliasing of the boundary of hemisphere, as in the former case). Aliasing antipodal points in second factor of (5), one gets $\mathbb{R}P^3$. Such an orbit is then a bundle

$$S^2 \coprod \mathbb{R}P^3, \tag{6}$$

where S^2 is the base, and $\mathbb{R}P^3$ is the fibre. This bundle is not trivial. Suppose it is. Then it is possible to define a continuous function from S^2 to S^1 (the global section of the bundle — see [2]). Such a section can define the global phase of first qubit (see (4)), which is impossible due to the nontriviality of the first Hopf bundle — see [6, 7] and references therein.

3.3. Maximaly entangled states

The stabiliser of this orbit has the form $U \times U^\dagger$, where $U \in \mathrm{SU}(2)$. It means, that every operation on the first qubit is equal to identity up to an element of the stabiliser. Deciding to represent the state of the first qubit, one has to choose a proper element of the stabiliser, and the representative of the state of the second qubit becomes fixed and unique. It gives $\mathbb{R}P^3$ as the manifold of this orbit.

The present section reproduces the results of [4], but without using Schmidt decomposition, which makes it possible for this method to be used in three qubit problem later on.

4. The Geometry of Orbits in Three Qubit Problem

4.1. Bystander states

This class of states, introduced by Carteret and Sudbery in [1], groups the states, where one of qubits is not entangled with the other two. Assume, that the first qubit is not entangled with the other two. Using again the results of [1], the stabiliser of such states can be written as $e^{i\alpha\sigma_z} \times U$, where U is an element of the stabiliser of two other qubits. The stabiliser is now the product of two subgroups, namely of the first qubit stabiliser and the second qubit one. It means, that the set of states is a product of two manifolds: $\mathrm{SU}(2) \,/\, \mathrm{U}(1)$ and one of the manifolds from the previous section. The result of aliasing in the first factor of this product was found in the previous section and it is the Bloch sphere of the first qubit. The geometry of an orbit of a bystander state is:

- $S^2 \times S^2 \times S^2$, when two other qubits are not entangled,
- $S^2 \times (S^2 \coprod \mathbb{R}P^3)$, when two other qubits are entangled, but not maximally entangled,
- $S^2 \times \mathbb{R}P^3$, when two other qubits are maximally entangled.

Because the first qubit is not entangled with the other two, the set of states of such system should be the product of the set of states of first qubit and the set of states of the other two, so it makes it easy to predict the form of the above manifolds.

4.2. GHZ class

The orbits from this class are represented by vector states of the form: $|\Psi\rangle = p|000\rangle + q|111\rangle$, where $p, q \in \mathbb{R}_+$. For $p = q$, $|\Psi\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$ is called GHZ (Greenberger, Horne, Zeilinger) state. This is a state which maximally violates Bell's inequality for three particles (see [9]). When the system is in GHZ state, all three qubits are entangled, but the mixed state of every pair, obtained by tracing out one of the qubits, is separable (see [10, 11, 12]). When $p \neq q$, the stabiliser of such orbit has the form

$$e^{i\theta\sigma_z} \times e^{i\alpha\sigma_z} \times e^{-i(\alpha+\theta)\sigma_z} \tag{7}$$

The elements of the form

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \times \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \cdot \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix}$$

$$\times \begin{bmatrix} e^{-i(\alpha+\theta)} & 0 \\ 0 & e^{i(\alpha+\theta)} \end{bmatrix} \cdot \begin{bmatrix} r & s \\ -\bar{s} & \bar{r} \end{bmatrix} = \begin{bmatrix} a \cdot e^{i\theta} & b \cdot e^{i\theta} \\ -\bar{b} \cdot e^{-i\theta} & \bar{a} \cdot e^{-i\theta} \end{bmatrix}$$

$$\times \begin{bmatrix} c \cdot e^{i\alpha} & d \cdot e^{i\alpha} \\ -\bar{d} \cdot e^{-i\alpha} & \bar{c} \cdot e^{-i\alpha} \end{bmatrix} \times \begin{bmatrix} r \cdot e^{-i(\alpha+\theta)} & r \cdot e^{-i(\alpha+\theta)} \\ -\bar{s} \cdot e^{i(\alpha+\theta)} & \bar{r} \cdot e^{i(\alpha+\theta)} \end{bmatrix}$$

represent the same operation for all α and θ. Taking $\alpha = -\arg a$ and $\theta = -\arg b$, after reparametrization

$$\begin{array}{ccc} a \cdot e^{i\alpha} & \to & a \\ b \cdot e^{i\alpha} & \to & b \\ c \cdot e^{i\theta} & \to & c \\ d \cdot e^{i\theta} & \to & d \\ r \cdot e^{-i(\alpha+\theta)} & \to & r \\ s \cdot e^{-i(\alpha+\theta)} & \to & s\,, \end{array} \tag{8}$$

one gets the unique (up to aliasing the boundary of hemispheres) representative of the form

$$\begin{bmatrix} a & b \\ -\bar{b} & a \end{bmatrix} \times \begin{bmatrix} c & d \\ -\bar{d} & c \end{bmatrix} \times \begin{bmatrix} r & s \\ -\bar{s} & \bar{r} \end{bmatrix}, \tag{9}$$

where $a, c \in \mathbb{R}_+$ and $b, d, r, s \in \mathbb{C}$. After aliasing the boundaries of hemispheres in two first factors, one obtains the manifold

$$(S^2 \times S^2) \coprod \mathbb{R}P^3, \tag{10}$$

where $S^2 \times S^2$ is the base, and $\mathbb{R}P^3$ is the fibre. Fixing a point on one sphere, one gets the bundle (6), which is not trivial, so the whole bundle cannot be trivial either.

When $p = q$ (GHZ state), a different subset of the stabiliser appears:

$$\begin{bmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{bmatrix} \times \begin{bmatrix} 0 & e^{i\alpha} \\ -e^{-i\alpha} & 0 \end{bmatrix} \times \begin{bmatrix} 0 & e^{-i(\theta+\alpha)} \\ -e^{i(\theta+\alpha)} & 0 \end{bmatrix}, \tag{11}$$

because the element

$$\sigma_y \times \sigma_y \times \sigma_y \tag{12}$$

is now in the stabiliser, and the subset (11) is the image of multiplying (7) by (12). The square of (12) yields identity, so this additional element in the stabiliser aliases the pairs of elements differing by (12). Acting by (12) on a representative of an element of the first factor of the base of (10), one obtains

$$\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \cdot \left[\begin{array}{cc} a & b \\ -\bar{b} & a \end{array}\right] = \left[\begin{array}{cc} -|b| \cdot e^{-i\arg b} & a \cdot e^{i\arg b} \cdot e^{-i\arg b} \\ -a \cdot e^{-i\arg b} \cdot e^{i\arg b} & -|b| \cdot e^{i\arg b} \end{array}\right]$$

$$= \left[\begin{array}{cc} e^{i(\arg b+\pi)} & 0 \\ 0 & e^{-i(\arg b+\pi)} \end{array}\right] \cdot \left[\begin{array}{cc} |b| & a \cdot e^{i(\arg b+\pi)} \\ -a \cdot e^{-i(\arg b+\pi)} & |b| \end{array}\right].$$

The same is obtained when the antipodal (on the Bloch sphere) element is used, so that antipodal elements on S^2 should be aliased. The same operation aliases the antipodal point in the second factor of (10). The antipodal points on $S^2 \times S^2$ (the base of (10)) should be aliased. A pair of points differing by a sign should also be aliased to a point. Such a point can be uniquely represented by the element of the pair with positive first coordinate in the first factor. The base of (10) in this case becomes the bundle

$$\mathbb{R}P^2 \coprod S^2,$$

where $\mathbb{R}P^2$ is the base, and S^2 is the fibre. Consider a loop which represents 1 in $\mathbb{Z}_2 = \Pi(\mathbb{R}P^2)$. Such a loop connects the antipodal points in S^3 — the universal covering of $\mathbb{R}P^2$. Fix an element in the fibre over the beginning point of the loop. Going along the loop, the fixed element changes to the anipodical, so the bundle is not trivial (see (1)). The third factor reparametrizes the fibre of (10):

$$(a,b) \;\to\; (-\bar{b},-\bar{a}) \;\sim\; (\bar{b},\bar{a})\,. \tag{13}$$

The manifold of GHZ state is

$$(\mathbb{R}P^2 \text{"}\times\text{"}\ S^2) \text{"}\times\text{"}\ \mathbb{R}P^3.$$

4.3. Slice ridge

There are also extended GHZ states [10]. Such orbits are represented by states of the form: $|\Psi\rangle = p|000\rangle + q|101\rangle + t|111\rangle$.

When p, q and t do not satisfy the condition

$$|p|^2 \ = \ |q|^2 + |t|^2, \tag{14}$$

then the stabiliser has the form

$$e^{i\theta\sigma_z} \times e^{-i\theta\sigma_z} \times I. \tag{15}$$

It means, that the set of operations

$$\begin{aligned} &\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \times \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \cdot \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r & s \\ -\bar{s} & \bar{r} \end{bmatrix} \\ &= \begin{bmatrix} a\cdot e^{i\theta} & b\cdot e^{i\theta} \\ -\bar{b}\cdot e^{-i\theta} & \bar{a}\cdot e^{-i\theta} \end{bmatrix} \times \begin{bmatrix} c\cdot e^{-i\theta} & d\cdot e^{-i\theta} \\ -\bar{d}\cdot e^{i\theta} & \bar{c}\cdot e^{i\theta} \end{bmatrix} \times \begin{bmatrix} r & s \\ -\bar{s} & -\bar{r} \end{bmatrix}, \end{aligned}$$

represents the same state for all θ. Choosing $\theta = -\arg a$ and then reparameterizing as

$$\begin{array}{rcl} a\cdot e^{i\theta} & \to & a \\ b\cdot e^{i\theta} & \to & b \\ c\cdot e^{-i\theta} & \to & c \\ d\cdot e^{-i\theta} & \to & d \end{array} \tag{16}$$

one gets the unique (again, up to aliasing the boundary points of hemisphere in the first factor) representation in the state of the form

$$\begin{bmatrix} a & b \\ -\bar{b} & a \end{bmatrix} \times \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix} \times \begin{bmatrix} r & s \\ -\bar{s} & \bar{r} \end{bmatrix}, \tag{17}$$

where $a \in \mathbb{R}_+$ and $b, c, d, r, s \in \mathbb{C}$. Such orbit is then

$$(S^2 \text{ "}\times\text{" } \mathbb{R}P^3) \times \mathbb{R}P^3, \tag{18}$$

where in the internal bundle S^2 is the base, and $\mathbb{R}P^3$ is the fibre. The last two equations in (16) establish an isomorphism between the fibre at a point of the base, and the typical fibre. The internal bundle is isomorphic with (5), so is not trivial.

When the condition (14) is satisfied, an additional subset of the stabiliser appears. Its elements are of the form

$$\begin{bmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{bmatrix} \times \begin{bmatrix} 0 & e^{-i(\theta+\chi)} \\ -e^{-i(\theta+\chi)} & 0 \end{bmatrix} \times \frac{-i}{p} \cdot \begin{bmatrix} |q| & \bar{t} \cdot e^{i\chi} \\ t \cdot e^{-i\chi} & |q| \end{bmatrix}, \tag{19}$$

where $\chi = \arg(q)$. The operation on the third qubit depends only on the orbit. Fixing an orbit and properly reparametrizing SU(2) groups of the second and the third qubit, (19) becomes

$$\begin{bmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{bmatrix} \times \begin{bmatrix} 0 & e^{-i\theta} \\ -e^{-i\theta} & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{20}$$

Changing θ, the parametrization of the third factor of (18) remains unchanged, so the cartesian product structure in (18) is conserved. It is easy to observe, that (20) is an image of (15) when multiplied by

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{21}$$

Acting with (21) on an element of (18) of the form (17) changes the point of the base of internal bundle in (18) to the antipodal one, and reparameterizes the fibre over this point. Such points in the base should be aliased (in the same way as in the GHZ-state problem), and the geometry of such orbit is

$$(\mathbb{R}P^2 \coprod \mathbb{R}P^3) \times \mathbb{R}P^3. \tag{22}$$

4.4. Beechnut states

The states of the form

$$|\Psi\rangle = p|000\rangle + q|101\rangle + t|110\rangle \tag{23}$$

are called beechnut states [1], and the orbits they represent have the stabiliser of the form

$$e^{i\phi\sigma_z} \times e^{i\phi\sigma_z} \times e^{-i\phi\sigma_z}. \tag{24}$$

It means, that all local operations of the form

$$\begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \times \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \cdot \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix} \times \begin{bmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{bmatrix} \cdot \begin{bmatrix} r & s \\ -\bar{s} & \bar{r} \end{bmatrix}$$
$$= \begin{bmatrix} a \cdot e^{i\phi} & b \cdot e^{i\phi} \\ -\bar{b} \cdot e^{-i\phi} & \bar{a} \cdot e^{-i\phi} \end{bmatrix} \times \begin{bmatrix} a \cdot e^{i\phi} & b \cdot e^{i\phi} \\ -\bar{b} \cdot e^{-i\phi} & \bar{a} \cdot e^{-i\phi} \end{bmatrix} \times \begin{bmatrix} a \cdot e^{-i\phi} & b \cdot e^{-i\phi} \\ -\bar{b} \cdot e^{i\phi} & \bar{a} \cdot e^{i\phi} \end{bmatrix}$$

represent the same state for all ϕ. Choosing $\phi = -\arg a$, after reparametrization

$$\begin{array}{rcl} a \cdot e^{i\phi} & \rightarrow & a \\ b \cdot e^{i\phi} & \rightarrow & b \\ c \cdot e^{i\phi} & \rightarrow & c \\ d \cdot e^{i\phi} & \rightarrow & d \\ r \cdot e^{-i\phi} & \rightarrow & r \\ s \cdot e^{-i\phi} & \rightarrow & s\,, \end{array} \tag{25}$$

one obtains the representative of the state in the form

$$\begin{bmatrix} a & b \\ -\bar{b} & a \end{bmatrix} \times \begin{bmatrix} c & d \\ -\bar{d} & \bar{c} \end{bmatrix} \times \begin{bmatrix} r & s \\ -\bar{s} & \bar{r} \end{bmatrix}, \tag{26}$$

where $a \in \mathbb{R}_+$ end $b, c, d, r, s \in \mathbb{C}_+$, and the boundary points in the first factor of (26) should be aliased. Hence the geometry of the orbit is

$$S^2 \coprod (\mathbb{R}P^3 \times \mathbb{R}P^3)\,, \tag{27}$$

where S^2 is the base, and $(\mathbb{R}P^3 \times \mathbb{R}P^3)$ is the fibre. The last four lines in (25) define an isomorphism between the fibre at a fixed point and the typical fibre.

Notice that the local operation

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

on the representative of the form (23) gives the state

$$|\Psi\rangle = p|100\rangle + q|001\rangle + t|010\rangle \,.$$

When

$$p = q = t = \frac{1}{\sqrt{3}}$$

then $|\Psi\rangle$ is the famous Werner state $|W\rangle$. When a 3-qubit system is in the Werner state, every two qubits are entangled, and the entanglement of the system consist of two-qubit entanglement only, in opposition to the GHZ state (see [10, 11, 12]).

5. Full-Dimensional Orbits

The generic orbit is an orbit of full dimension, so that the stabiliser is discrete. Now the set of solutions of the equation

$$U \times V \times W|\Psi\rangle \;=\; e^{i\phi}|\Psi\rangle \tag{28}$$

in $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ forms a discrete subgroup. Because this is a discrete set of solutions of a polynomial equation, it is finite. Taking the state $|\Psi\rangle$ in the form

$$|\Psi\rangle \;=\; \sum_{i,j,k=0}^{1} t_{ijk}|ijk\rangle \,, \tag{29}$$

(28) can be written as

$$t_{ijk} \;=\; e^{i\alpha} \sum_{l,m,n=0}^{1} u_{il} v_{jm} w_{kn} t_{lmn} \,. \tag{30}$$

Every state vector (29) can be written in the form [10]

$$|\Psi\rangle \;=\; \lambda_0|000\rangle+\lambda_1e^{i\alpha}|100\rangle+\lambda_2|101\rangle+\lambda_3|110\rangle+\lambda_4|111\rangle \tag{31}$$

where

$$\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4\geq 0\,,$$
$$\lambda_0+\lambda_1+\lambda_2+\lambda_3+\lambda_4=1\,,$$
$$\phi\in[0,\pi]$$

using local operations. Only vector states of this form will be used from now on.

5.1. Generic states

One can split the $2\times 2\times 2$ matrix $[t_{ijk}]$ from (29), (30)[1] with respect to the first qubit and then (30) turns into two matrix equations

$$\begin{aligned} V\Big(u_{11}\begin{bmatrix}\lambda_0 & 0\\ 0 & 0\end{bmatrix}+u_{12}\begin{bmatrix}\lambda_1e^{i\phi} & \lambda_2\\ \lambda_3 & \lambda_4\end{bmatrix}\Big)W^T &= e^{i\alpha}\begin{bmatrix}\lambda_0 & 0\\ 0 & 0\end{bmatrix},\\ V\Big(-\bar{u}_{12}\begin{bmatrix}\lambda_0 & 0\\ 0 & 0\end{bmatrix}+\bar{u}_{11}\begin{bmatrix}\lambda_1e^{i\phi} & \lambda_2\\ \lambda_3 & \lambda_4\end{bmatrix}\Big)W^T &= e^{i\alpha}\begin{bmatrix}\lambda_1e^{i\phi} & \lambda_2\\ \lambda_3 & \lambda_4\end{bmatrix}. \end{aligned} \tag{32}$$

Taking the determinants of the above, one gets two equations:

$$u_{11}u_{12}(\lambda_0\lambda_4)+u_{12}^2(\lambda_1\lambda_4e^{i\phi}-\lambda_3\lambda_2) \;=\; 0\,, \tag{33}$$

$$-\bar{u}_{11}\bar{u}_{12}(\lambda_0\lambda_4)+\bar{u}_{11}^2(\lambda_1\lambda_4e^{i\phi}-\lambda_3\lambda_2) \;=\; e^{i\alpha}(\lambda_1\lambda_4e^{i\phi}-\lambda_3\lambda_2)\,. \tag{34}$$

When $\lambda_0=0$, the state belongs to the bystander states, so only the states with $\lambda_0\neq 0$ will be considered now. When $(\lambda_1\lambda_4e^{i\phi}-\lambda_3\lambda_2)=0$ (assumption **Gen1** in Theorem 1 of [1] fails) then the state belongs to the GHZ class or slice ridge, with the exclusion of the case when $\lambda_1,\lambda_2,\lambda_3,\lambda_4\neq 0$. This class is called *semigeneric states* and will be considered in the next subsection. When $\lambda_4=0$,

[1] The theory of such matrices and its relationship with the theory of entanglement were considered in [13].

then assumption **Gen2** in Theorem 1 of [1] fails and the state belongs to the beechnut states considered in the previous section.

Consider now the situation when

$$\begin{aligned} A &\stackrel{\text{df}}{=} \lambda_0\lambda_4 \neq 0\,, \\ B &\stackrel{\text{df}}{=} \lambda_1\lambda_4 e^{i\phi} - \lambda_3\lambda_2 \neq 0\,. \end{aligned}$$

The number u_{11} cannot be zero. If $u_{11} = 0$, then (33) takes the form

$$u_{12}^2(\lambda_1\lambda_4 e^{i\phi} - \lambda_3\lambda_2) = 0\,,$$

which implies $u_{12} = 0$ — a contradiction with SU(2) condition $|u_{11}|^2 + |u_{12}|^2 = 1$.

When $u_{12} = 0$, then (34) implies $u_{11} = e^{-i\alpha/2}$. The solution is of the form

$$U = \begin{bmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{bmatrix}. \tag{35}$$

When $u_{11}, u_{12} \neq 0$, then the first equation divided by u_{12} and the condition $|u_{11}|^2 + |u_{12}|^2 = 1$ yields two equations

$$\begin{aligned} u_{11}A + u_{12}B &= 0\,, \\ |u_{11}|^2 + |u_{12}|^2 &= 1\,, \end{aligned} \tag{36}$$

which give the solution

$$U = \frac{1}{\sqrt{A^2 + |B|^2}} \begin{bmatrix} -Be^{i\beta} & Ae^{i\beta} \\ -Ae^{-i\beta} & -\bar{B}e^{i\beta} \end{bmatrix}. \tag{37}$$

Because the set of solutions should have a group structure, the second powers of (35), (37) should be either solutions, or the identity in SU(2). This condition implies that $\alpha = \pm\pi$ and $\beta = \pm\pi/2 - \arg B$.

The group of solutions is

$$G_u = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \frac{i}{\sqrt{A^2+|B|^2}} \begin{bmatrix} -|B| & Ae^{-i\arg B} \\ Ae^{i\arg B} & |B| \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \frac{-i}{\sqrt{A^2+|B|^2}} \begin{bmatrix} -|B| & Ae^{-i\arg B} \\ Ae^{i\arg B} & |B| \end{bmatrix} \right\}.$$

This group is generated by the element

$$\frac{i}{\sqrt{A^2+|B|^2}} \begin{bmatrix} -|B| & Ae^{-i\arg B} \\ Ae^{i\arg B} & |B| \end{bmatrix}$$

of rank 4, so it is isomorphic to $\mathbb{Z}_4$. This is the maximal group whose elements satisfy (33) and (34).

Splitting the $2 \times 2 \times 2$ matrix $[t_{ijk}]$ from (29), (30) into two 2×2 matrixes, with respect to the second qubit, the equation (30) changes into two equations

$$\begin{aligned} U\Big(v_{11} \begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_2 \end{bmatrix} + v_{12} \begin{bmatrix} 0 & 0 \\ \lambda_3 & \lambda_4 \end{bmatrix}\Big) W^T &= e^{i\alpha} \begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_2 \end{bmatrix}, \\ U\Big(-\bar{v}_{12} \begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_2 \end{bmatrix} + \bar{v}_{11} \begin{bmatrix} 0 & 0 \\ \lambda_3 & \lambda_4 \end{bmatrix}\Big) W^T &= e^{i\alpha} \begin{bmatrix} 0 & 0 \\ \lambda_3 & \lambda_4 \end{bmatrix}. \end{aligned} \tag{38}$$

Taking the determinant of above yields

$$\begin{aligned} v_{11}\lambda_0(v_{11}\lambda_2 + v_{12}\lambda_4) &= e^{i\alpha}\lambda_0\lambda_2\,, \\ -\bar{v}_{12}\lambda_0(-\bar{v}_{12}\lambda_2 + \bar{v}_{11}\lambda_4) &= 0\,. \end{aligned} \tag{39}$$

When $\lambda_0 = 0$, then the set belongs to the bystander states, so consider $\lambda_0 \neq 0$. When $\lambda_4 = 0$, then the state belongs to the beechnut states (see above). The situation, when $\lambda_2 = 0$ (semigeneric states) will be considered later. The situation when $\lambda_0, \lambda_4, \lambda_2 \neq 0$ is considered now.

The number $v_{11} \neq 0$ for the same reason as the number $u_{11} \neq 0$ (see above).

When $v_{12} = 0$, the solution of (39) is

$$V = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix} \tag{40}$$

similarly as for U in (35).

When $v_{11}, v_{12} \neq 0$, then the two equations

$$\begin{aligned} \bar{v}_{12}\lambda_2 - \bar{v}_{11}\lambda_4 &= 0\,, \\ |v_{11}|^2 + |v_{12}|^2 &= 1 \end{aligned} \tag{41}$$

(analogous to (36)) give the solution

$$V = \frac{1}{\sqrt{\lambda_2^2+\lambda_4^2}} \begin{bmatrix} \lambda_2 e^{i\beta} & \lambda_4 e^{i\beta} \\ -\lambda_4 e^{-i\beta} & \lambda_2 e^{-i\beta} \end{bmatrix}. \tag{42}$$

The group condition for the set of solutions of (39) gives $\alpha = \pm\pi$ in (40) and $\beta = \pm\pi/2$ in (42). The set of solutions of (39) is then the group

$$G_v = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \frac{i}{\sqrt{\lambda_2^2+\lambda_4^2}} \begin{bmatrix} \lambda_2 & \lambda_4 \\ \lambda_4 & -\lambda_2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \frac{-i}{\sqrt{\lambda_2^2+\lambda_4^2}} \begin{bmatrix} \lambda_2 & \lambda_4 \\ \lambda_4 & -\lambda_2 \end{bmatrix} \right\},$$

generated by the element

$$\frac{i}{\sqrt{\lambda_2^2+\lambda_4^2}} \begin{bmatrix} \lambda_2 & \lambda_4 \\ \lambda_4 & -\lambda_2 \end{bmatrix}$$

of rank 4. This group is isomorphic to $\mathbb{Z}_4$ and is the maximal group contained in the set of solutions of (39).

Splitting the $2 \times 2 \times 2$ matrix $[t_{ijk}]$ from (29), (30) into two 2×2 matrices, with respect to the third qubit, equation (30) changes into

$$\begin{aligned} U\Big(w_{11} \begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_3 \end{bmatrix} + w_{12} \begin{bmatrix} 0 & 0 \\ \lambda_2 & \lambda_4 \end{bmatrix}\Big) V^T &= e^{i\alpha} \begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_3 \end{bmatrix} \\ U\Big(-\bar{w}_{12} \begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_3 \end{bmatrix} + \bar{w}_{11} \begin{bmatrix} 0 & 0 \\ \lambda_2 & \lambda_4 \end{bmatrix}\Big) V^T &= e^{i\alpha} \begin{bmatrix} 0 & 0 \\ \lambda_2 & \lambda_4 \end{bmatrix}. \end{aligned} \tag{43}$$

Taking the determinants of above equations one obtains now

$$\begin{aligned} w_{11}\lambda_0(w_{11}\lambda_3 + w_{12}\lambda_4) &= e^{i\alpha}\lambda_0\lambda_3 \\ -\bar{w}_{12}\lambda_0(-\bar{w}_{12}\lambda_3 + \bar{w}_{11}\lambda_4) &= 0\,. \end{aligned} \tag{44}$$

When $\lambda_0 = 0$, then the set belongs to the bystander states. When $\lambda_4 = 0$, then the state belongs to the beechnut states (see above). The situation when $\lambda_3 = 0$ (semigeneric states) will be considered later. The situation when $\lambda_0, \lambda_4, \lambda_3 \neq 0$ is considered now.

The number $w_{11} \neq 0$ for the same reason as for the numbers $u_{11}, v_{11} \neq 0$ (see above).

When $w_{12} = 0$, the solution of (43) is the matrix

$$W = \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}, \tag{45}$$

similarly as for U and V above.

When $w_{11}, w_{12} \neq 0$, then the equations

$$\begin{aligned} \bar{w}_{12}\lambda_2 - \bar{w}_{11}\lambda_4 &= 0\,, \\ |w_{11}|^2 + |w_{12}|^2 &= 1 \end{aligned} \tag{46}$$

(analogous to (36) and (41)) give the solution

$$V = \frac{1}{\sqrt{\lambda_3^2 + \lambda_4^2}} \begin{bmatrix} \lambda_3 e^{i\beta} & \lambda_4 e^{i\beta} \\ -\lambda_4 e^{-i\beta} & \lambda_3 e^{-i\beta} \end{bmatrix}. \tag{47}$$

The group condition for the set of solutions of (44) gives $\alpha = \pm\pi$ in (45) and $\beta = \pm\pi/2$ in (47). The set of solutions of (44) is then the group

$$G_w = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \frac{i}{\sqrt{\lambda_3^2 + \lambda_4^2}} \begin{bmatrix} \lambda_3 & \lambda_4 \\ \lambda_4 & -\lambda_3 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \frac{-i}{\sqrt{\lambda_3^2 + \lambda_4^2}} \begin{bmatrix} \lambda_3 & \lambda_4 \\ \lambda_4 & -\lambda_3 \end{bmatrix} \right\},$$

generated by the element

$$\frac{i}{\sqrt{\lambda_3^2 + \lambda_4^2}} \begin{bmatrix} \lambda_3 & \lambda_4 \\ \lambda_4 & -\lambda_3 \end{bmatrix}$$

of rank 4. This group is isomorphic to $\mathbb{Z}_4$ and is the maximal group contained in set of solutions of (44).

Equations (34), (39), (44) are more general than equations (32), (38), (43), so set of solutions of (32), (38), (43) (group) is contained in set of solutions of (34), (39), (44), which is not a group in general, as it has been shown above. The maximal group satisfying equations (34), (39), (44) is $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. The solution of (32), (38), (43) is then a subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.

The group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ satisfies the stabiliser equation (28) in an obvious way (each operation changes the global phase by π). This subgroup divides the group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ into eight cosets ($64/8 = 8$ — Lagrange's theorem) represented by elements:

$$(0,0,0) \quad (1,0,0) \quad (0,1,0) \quad (0,0,1) \quad (0,1,1) \quad (1,0,1) \quad (1,1,0) \quad (1,1,1)\,. \tag{48}$$

Now, all the cosets except for the first one will be excluded by substituting the elements representing them into the stabiliser equations.

Consider the element $(1,0,0)$, and substitute it in the first equation of (32):

$$\frac{-i|B|}{\sqrt{|B|^2+|A|^2}}\begin{bmatrix} \lambda_0 & 0 \\ 0 & 0 \end{bmatrix} + \frac{iAe^{-i\arg B}}{\sqrt{|B|^2+|A|^2}}\begin{bmatrix} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} =$$
$$= \frac{i}{\sqrt{|B|^2+A^2}}\begin{bmatrix} -|B|\lambda_0 + Ae^{-i\arg B}\lambda_1 e^{i\phi} & Ae^{-i\arg B}\lambda_2 \\ Ae^{-i\arg B}\lambda_3 & Ae^{-i\arg B}\lambda_4 \end{bmatrix} = e^{i\alpha}\begin{bmatrix} \lambda_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Because $A, \lambda_4 > 0$ by assumption (if $A = \lambda_0\lambda_4 = 0$, then the state belongs to the beechnut states), the element $(1,0,0)$ is not in the stabilizer, so it excludes the coset represented by this element.

Consider the element $(0,1,0)$, and substitute it in the second equation of (38),

$$\frac{i\lambda_4}{\sqrt{\lambda_2^2+\lambda_4^2}}\begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_2 \end{bmatrix} - \frac{i\lambda_2}{\sqrt{\lambda_2^2+\lambda_4^2}}\begin{bmatrix} 0 & 0 \\ \lambda_3 & \lambda_4 \end{bmatrix} =$$
$$= \frac{i}{\sqrt{\lambda_2^2+\lambda_4^2}}\begin{bmatrix} \lambda_0\lambda_4 & 0 \\ \lambda_1\lambda_4 e^{i\phi} - \lambda_3\lambda_2 & 0 \end{bmatrix} = e^{i\alpha}\begin{bmatrix} 0 & 0 \\ \lambda_3 & \lambda_4 \end{bmatrix}.$$

Because $\lambda_0, \lambda_4 > 0$ by assumption (if not, the state belongs to the beechnut states), the element $(0, 0, 1)$ is not in the stabilizer, so it excludes the coset represented by this element.

Likewise consider the element $(0, 0, 1)$, and substitute it in the second equation of (43)

$$\frac{i\lambda_4}{\sqrt{\lambda_3^2+\lambda_4^2}}\begin{bmatrix}\lambda_0 & 0\\ \lambda_1 e^{i\phi} & \lambda_3\end{bmatrix} - \frac{i\lambda_3}{\sqrt{\lambda_3^2+\lambda_4^2}}\begin{bmatrix}0 & 0\\ \lambda_2 & \lambda_4\end{bmatrix} = \\ = \frac{i}{\sqrt{\lambda_3^2+\lambda_4^2}}\begin{bmatrix}\lambda_0\lambda_4 & 0\\ \lambda_1\lambda_4 e^{i\phi} - \lambda_2\lambda_3 & 0\end{bmatrix} = e^{i\alpha}\begin{bmatrix}0 & 0\\ \lambda_2 & \lambda_4\end{bmatrix}.$$

Again since $\lambda_0, \lambda_4 > 0$ by assumption, the element $(0, 0, 1)$ is not in the stabilizer, so it excludes the coset represented by this element.

Similarly, substituting $(0, 1, 1)$, in the first equation of (32),

$$\frac{i}{\sqrt{\lambda_2^2+\lambda_4^2}}\begin{bmatrix}\lambda_2 & \lambda_4\\ \lambda_4 & -\lambda_2\end{bmatrix}\begin{bmatrix}\lambda_0 & 0\\ 0 & 0\end{bmatrix}\frac{i}{\sqrt{\lambda_3^2+\lambda_4^2}}\begin{bmatrix}\lambda_3 & \lambda_4\\ \lambda_4 & -\lambda_3\end{bmatrix} = \\ = \frac{i\lambda_0}{\sqrt{\lambda_2^2+\lambda_4^2}\sqrt{\lambda_3^2+\lambda_4^2}}\begin{bmatrix}\lambda_2\lambda_3 & \lambda_2\lambda_4\\ \lambda_4\lambda_3 & \lambda_4^2\end{bmatrix} = e^{i\alpha}\begin{bmatrix}\lambda_0 & 0\\ 0 & 0\end{bmatrix}.$$

Leads to exclusion of the coset represented by this element since again $\lambda_0, \lambda_4 > 0$ by assumption.

Repeating the same way of reasoning for $(1, 0, 1)$, and substituting it in the second equation of (38),

$$\frac{i}{\sqrt{|B|^2+A^2}}\begin{bmatrix}-|B| & Ae^{-i\arg B}\\ Ae^{i\arg B} & |B|\end{bmatrix}\begin{bmatrix}0 & 0\\ \lambda_3 & \lambda_4\end{bmatrix}\frac{i}{\sqrt{\lambda_3^2+\lambda_4^2}}\begin{bmatrix}\lambda_3 & \lambda_4\\ \lambda_4 & -\lambda_3\end{bmatrix} = \\ = -\frac{\sqrt{\lambda_3^2+\lambda_4^2}}{\sqrt{|B|^2+A^2}}\begin{bmatrix}Ae^{-i\arg B} & 0\\ |B| & 0\end{bmatrix} = e^{i\alpha}\begin{bmatrix}0 & 0\\ \lambda_3 & \lambda_4\end{bmatrix}$$

The asumption that $\lambda_4 > 0$ excludes the coset represented by this element.

Next $(1, 1, 0)$, is substituting in the second equation of (43),

$$\frac{i}{\sqrt{|B|^2+A^2}}\begin{bmatrix}-|B| & Ae^{-i\arg B}\\ Ae^{i\arg B} & |B|\end{bmatrix}\begin{bmatrix}0 & 0\\ \lambda_2 & \lambda_4\end{bmatrix}\frac{i}{\sqrt{\lambda_2^2+\lambda_4^2}}\begin{bmatrix}\lambda_2 & \lambda_4\\ \lambda_4 & -\lambda_2\end{bmatrix} =$$

$$= -\frac{\sqrt{\lambda_2^2+\lambda_4^2}}{\sqrt{|B|^2+A^2}}\left[\begin{array}{cc} Ae^{-i\arg B} & 0 \\ |B| & 0 \end{array}\right] = e^{i\alpha}\left[\begin{array}{cc} 0 & 0 \\ \lambda_2 & \lambda_4 \end{array}\right]$$

Once again $\lambda_4 > 0$ excludes the coset represented by $(1, 1, 0)$

Finally, consider the element $(1, 1, 1)$, and plug it in the first equation of (32).

$$\frac{i}{\sqrt{\lambda_2^2+\lambda_4^2}}\left[\begin{array}{cc} \lambda_2 & \lambda_4 \\ \lambda_4 & -\lambda_2 \end{array}\right]\Big(\frac{-i|B|}{\sqrt{|B|^2+|A|^2}}\left[\begin{array}{cc} \lambda_0 & 0 \\ 0 & 0 \end{array}\right] +$$
$$+\frac{iAe^{-i\arg B}}{\sqrt{|B|^2+|A|^2}}\left[\begin{array}{cc} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{array}\right]\Big)\frac{i}{\sqrt{\lambda_3^2+\lambda_4^2}}\left[\begin{array}{cc} \lambda_3 & \lambda_4 \\ \lambda_4 & -\lambda_3 \end{array}\right] =$$
$$= \frac{i}{\sqrt{\lambda_2^2+\lambda_4^2}}\left[\begin{array}{cc} \lambda_2 & \lambda_4 \\ \lambda_4 & -\lambda_2 \end{array}\right] \times$$
$$\times \frac{i}{\sqrt{|B|^2+A^2}}\left[\begin{array}{cc} -|B|\lambda_0 + Ae^{-i\arg B}\lambda_1 e^{i\phi} & Ae^{-i\arg B}\lambda_2 \\ Ae^{-i\arg B}\lambda_3 & Ae^{-i\arg B}\lambda_4 \end{array}\right] \times$$
$$\times \frac{i}{\sqrt{\lambda_3^2+\lambda_4^2}}\left[\begin{array}{cc} \lambda_3 & \lambda_4 \\ \lambda_4 & -\lambda_3 \end{array}\right] = e^{i\alpha}\left[\begin{array}{cc} \lambda_0 & 0 \\ 0 & 0 \end{array}\right],$$

which implies

$$\frac{i}{\sqrt{|B|^2+A^2}}\left[\begin{array}{cc} -|B|\lambda_0 + Ae^{-i\arg B}\lambda_1 e^{i\phi} & Ae^{-i\arg B}\lambda_2 \\ Ae^{-i\arg B}\lambda_3 & Ae^{-i\arg B}\lambda_4 \end{array}\right] =$$
$$= \frac{-i}{\sqrt{\lambda_2^2+\lambda_4^2}}\left[\begin{array}{cc} \lambda_2 & \lambda_4 \\ \lambda_4 & -\lambda_2 \end{array}\right] e^{i\alpha}\left[\begin{array}{cc} \lambda_0 & 0 \\ 0 & 0 \end{array}\right]\frac{-i}{\sqrt{\lambda_3^2+\lambda_4^2}}\left[\begin{array}{cc} \lambda_3 & \lambda_4 \\ \lambda_4 & -\lambda_3 \end{array}\right] =$$
$$= \frac{-\lambda_0 e^{i\alpha}}{\sqrt{\lambda_2^2+\lambda_4^2}\sqrt{\lambda_3^2+\lambda_4^2}}\left[\begin{array}{cc} \lambda_2\lambda_3 & \lambda_2\lambda_4 \\ \lambda_4\lambda_3 & \lambda_4^2 \end{array}\right].$$

It yields two equations

$$\begin{aligned} \frac{iA}{\sqrt{|B|^2+A^2}} &= \frac{-\lambda_0\lambda_4 e^{i(\alpha+\arg B)}}{\sqrt{\lambda_2^2+\lambda_4^2}\sqrt{\lambda_3^2+\lambda_4^2}} \\ \frac{iA}{\sqrt{|B|^2+A^2}}\Big(-\frac{|B|}{A}\lambda_0 + \lambda_1 e^{i\phi}\Big) &= \frac{-\lambda_0\lambda_2\lambda_3 e^{i(\alpha+\arg B)}}{\sqrt{\lambda_2^2+\lambda_4^2}\sqrt{\lambda_3^2+\lambda_4^2}} \end{aligned} \tag{49}$$

Dividing the second equation by the first (it is allowed because of the assumptions on the positivity of A and lambdas) one obtains

$$B\lambda_0 = \lambda_1\lambda_4 - \lambda_2\lambda_3 = B,$$

so $\lambda_0 = 1$. Normalization of the state gives the condition $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$, so the above result implies $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. The contradiction with assumptions of the positivity of lambdas excludes the last coset.

Please correct

The stabilizer of a generic state (in SU(2) × SU(2) × SU(2)) is

$$S^3/\mathbb{Z}_2 \times S^3/\mathbb{Z}_2 \times S^3/\mathbb{Z}_2 \;=\; \mathbb{R}P^3 \times \mathbb{R}P^3 \times \mathbb{R}P^3,$$

so the group of local operations acts freely on such orbits.

5.2. Semigeneric states

Consider again (32),

$$\begin{aligned} V\Big(u_{11}\begin{bmatrix} \lambda_0 & 0 \\ 0 & 0 \end{bmatrix} + u_{12}\begin{bmatrix} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}\Big)W^T &= e^{i\alpha}\begin{bmatrix} \lambda_0 & 0 \\ 0 & 0 \end{bmatrix} \\ V\Big(-\bar{u}_{12}\begin{bmatrix} \lambda_0 & 0 \\ 0 & 0 \end{bmatrix} + \bar{u}_{11}\begin{bmatrix} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}\Big)W^T &= e^{i\alpha}\begin{bmatrix} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}. \end{aligned} \tag{50}$$

Assume that

$$\det\begin{bmatrix} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix} = 0$$

and $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \neq 0$. Then the state belongs to the semigeneric states and has a discrete stabilizer (see [1]). Taking the determinant of (32) (recalled above as (50)) gives now only one equation

$$u_{11}u_{12}\lambda_0\lambda_4 \;=\; 0\,,$$

which has two solutions

$$U_1 \;=\; \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{bmatrix}, \qquad U_2 \;=\; \begin{bmatrix} 0 & e^{i\beta} \\ -e^{-i\beta} & 0 \end{bmatrix}. \tag{51}$$

Consider now the first possibility. Substituting such U_1 in the first equation of (32) one obtains:

$$\left[\begin{array}{cc} v_{11}w_{11} & v_{12}w_{12} \\ -\bar{v}_{12}w_{11} & -\bar{v}_{12}w_{12} \end{array}\right] = e^{i(\alpha-\beta)}\left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right]. \tag{52}$$

The solution of this equation is

$$V = \left[\begin{array}{cc} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{array}\right], \qquad W = \left[\begin{array}{cc} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{array}\right],$$

where the angles $\alpha, \beta, \gamma, \delta$ fulfill the condition: $\alpha - \beta = \gamma + \delta$. Substituting U, V and W in the second equation of (32) produces four equations for the angles:

$$\begin{aligned} \gamma + \delta &= \alpha + \beta\,, \\ \gamma - \delta &= \alpha + \beta\,, \\ -\gamma + \delta &= \alpha + \beta\,, \\ -\gamma - \delta &= \alpha + \beta\,, \end{aligned}$$

implying $\beta, \gamma, \delta \in \{0, \pi\}$. The solution of (32) is then

$$U = \pm\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \qquad V = \pm\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right], \qquad W = \pm\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]. \tag{53}$$

Consider now the second possibility of (51). Substituting U_2 in (32) one obtains

$$\begin{aligned} e^{i\beta}\left[\begin{array}{cc} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{array}\right] &= \lambda_0 e^{i\alpha}\left[\begin{array}{cc} \bar{v}_{11}\bar{w}_{11} & \bar{v}_{11}\bar{w}_{12} \\ \bar{v}_{12}\bar{w}_{11} & \bar{v}_{12}\bar{w}_{12} \end{array}\right] \\ -\lambda_0 e^{i\beta}\left[\begin{array}{cc} v_{11}w_{11} & -v_{11}\bar{w}_{12} \\ -\bar{v}_{12}w_{11} & \bar{v}_{12}\bar{w}_{12} \end{array}\right] &= e^{i\alpha}\left[\begin{array}{cc} \lambda_1 e^{i\phi} & \lambda_2 \\ \lambda_3 & \lambda_4 \end{array}\right]. \end{aligned} \tag{54}$$

The solution of (54) exists iff $\phi \in \{0, \pi\}$, and then it is

$$U = \pm\left[\begin{array}{cc} 0 & i \\ -i & 0 \end{array}\right],$$

$$V = \frac{\pm i}{\sqrt{\lambda_2^2 + \lambda_4^2}}\left[\begin{array}{cc} \lambda_2 & \lambda_4 \\ \lambda_4 & -\lambda_2 \end{array}\right], \qquad V = \frac{\pm i}{\sqrt{\lambda_1^2 + \lambda_2^2}}\left[\begin{array}{cc} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{array}\right]. \tag{55}$$

The typical orbit of semigeneric state (when $\phi \in (0,\pi)$) has the stabiliser of the form (53) — the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the orbit of such state is a trivial bundle:

$$S^3/\mathbb{Z}_2 \times S^3/\mathbb{Z}_2 \times S^3/\mathbb{Z}_2 \;=\; \mathbb{R}P^3 \times \mathbb{R}P^3 \times \mathbb{R}P^3.$$

When $\phi \in \{0,\pi\}$ (so when it is possible to represent the state by state vector with real coefficients), another subset (55) of the stabiliser appears. Then the stabiliser group is given by (53) and (55) — the group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. The orbit of such special state is

$$S^3/\mathbb{Z}_4 \times S^3/\mathbb{Z}_4 \times S^3/\mathbb{Z}_4$$

When the $2 \times 2 \times 2$ matrix $[t_{ijk}]$ of (29), (30) is divided with respect to the second qubit, then the state is described by two matrices

$$T_1 \;=\; \begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_2 \end{bmatrix}, \qquad T_2 \;=\; \begin{bmatrix} 0 & 0 \\ \lambda_3 & \lambda_4 \end{bmatrix}.$$

Suppose that $\lambda_2 = 0$, but the rest of the lambdas are positive (see the assumptions to (39)). Acting on the state by the following unitary operation in the third-qubit space

$$\frac{1}{\sqrt{\lambda_3^2+\lambda_4^2}} \begin{bmatrix} \lambda_3 & -\lambda_4 \\ \lambda_4 & \lambda_3 \end{bmatrix},$$

the matrices above take the form

$$T_1 \;=\; \frac{1}{\sqrt{\lambda_3+\lambda_4}} \begin{bmatrix} \lambda_0\lambda_3 & -\lambda_0\lambda_4 \\ \lambda_1\lambda_3 e^{i\phi} & -\lambda_1\lambda_4 \end{bmatrix}, \qquad T_2 = \sqrt{\lambda_3+\lambda_4} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now acting in the spaces of the first and the second qubit by operations

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(such operation relabels the base) and exchanging the labels of particles 1 and 2 one gets the same situation as above (when $[t_{ijk}]$ has been split with respect to the first qubit).

When the $2 \times 2 \times 2$ matrix $[t_{ijk}]$ of (29), (30) is divided with respect to the third qubit, then the state is described by two matrices

$$T_1 = \begin{bmatrix} \lambda_0 & 0 \\ \lambda_1 e^{i\phi} & \lambda_3 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0 & 0 \\ \lambda_2 & \lambda_4 \end{bmatrix}.$$

Suppose that $\lambda_3 = 0$, but the rest of the lambdas are positive (see the assumptions to (44)). Acting on the state by the following unitary operation in second-qubit space

$$\frac{1}{\sqrt{\lambda_2^2 + \lambda_4^2}} \begin{bmatrix} \lambda_2 & -\lambda_4 \\ \lambda_4 & \lambda_2 \end{bmatrix},$$

the matrices above take the form

$$T_1 = \frac{1}{\sqrt{\lambda_2 + \lambda_4}} \begin{bmatrix} \lambda_0\lambda_2 & -\lambda_0\lambda_4 \\ \lambda_1\lambda_2 e^{i\phi} & -\lambda_1\lambda_4 \end{bmatrix}, \qquad T_2 = \sqrt{\lambda_2 + \lambda_4} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now acting in the spaces of the first and the third qubit by operations

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and exchanging the labels of particles 1 and 3 one gets the same situation as above (when $[t_{ijk}]$ has been split with respect to the first qubit).

6. Remark

As mentioned in Sect. 3, lower-dimensional orbits have extreme values of concurrence. In the three-qubit case there are five local invariants [14]. As in the two-qubit case, the nongeneric orbits have extreme values of some of these invariants (see [1]). Their values were found in [10], and the subset of non-generic orbits was embedded in the five-dimensional set of orbits parameterized (not uniquely) by a canonical form of an orbit. These subsets lie in the boundary of set of orbits.

The results of this work generalize the above fact. Orbits of full dimension have a trivial stabiliser (antipodal points in S^3 represent the same unitary operation!), excluding the case of semigeneric states with real coefficients. But orbits of such states are in the boundary of the set of orbits ($\phi \in \{0, \pi\}$). So the subset of orbits with nontrivial stabiliser is contained in the boundary of the set of orbits.

Bibliography

[1] H. A. Carteret, A. Sudbery, *Local symmetry properties of pure three-qubit states*, J. Phys. A **33**, 4981 (2000).

[2] N. E. Steenrod, *The topology of fibre bundles*, Princeton University Press, 1951.

[3] A. Dold, *Lectures on algebraic topology*, Springer-Verlag, 1972.

[4] M. M. Sinołęcka, K. Życzkowski, M. Kuś, *Manifolds of interconvertible pure states*, Act. Phys. Pol. B **33**, 2081 (2002).

[5] A. Peres, *Higher order Schmidt decompositions*, Phys. Lett A **202**, 16 (1995).

[6] R. Moseri, R. Dandoloff, *Geometry of entangled states, Bloch spheres and Hopf fibrations*, J. Phys. A: Math Gen. **34**, 10243 (2001).

[7] B. A. Bernevig, H. D. Chen, *Geometry of three-qubit state, entanglement and division algebras*, J. Phys. A: Math. Gen. **36**, 8325 (2003).

[8] M. Kuś, K. Życzkowski, *Geometry of entangled states*, Phys. Rev. A **63**, 032307 (2001).

[9] D. M. Greenberger, M. A. Horne, A. Shimony, A. Zeilinger, *Bell's theorem without inequalities*, Am. J. Phys. **58**(12), 1131 (1990).

[10] A. Acin, A. Adrianov, L. Costa, E. Jané, J. I. Latorre, R. Tarrach, *Generalized Schmidt decomposition and classification of three-quantum-bit states*, Phys. Rev. Lett. **85**, 1560 (2000).

[11] V. Coffman, J. Kundu, W. K. Wooters, *Distributed Entanglement*, Phys. Rev. A **61**, 052306 (2000).

[12] W.Dür, G. Vidal, J. I. Cirac, *Three qubits can be entangled in two inequivalent ways*, Phys. Rev A **62**, 062314 (2000).

[13] A. Miyake, *Classification of multipartite entangled states by multidimensional determinants*, Phys. Rev. A **67**, 012108 (2003).

[14] A. Sudbery, *On local invariants of pure three-qubit states*, J. Phys. A **34**, 643 (2001).