Admissible Partitions and the expansion of the square of the Vandermonde determinant in N variables

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Abstract. Notes on the expansion of powers of the Vandermonde determinant.

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1. Introduction

The Vandermonde alternating function in N variables is defined as

$$V(z_1,\ldots,z_N)=\prod_{i<j}^{N}(z_i-z_j). \tag{1-1}$$

The even powers of V are symmetric functions and may be expanded as a sum of Schur functions

$$s_\lambda(z_1,\ldots,z_N)=\{\lambda\}=\{\lambda_1,\ldots,\lambda_p\} \tag{1-2}$$

In these notes we shall limit ourselves to just the second power and write

$$V_N^2=\sum_{\lambda\vdash n}c^\lambda\{\lambda\} \tag{1-3}$$

The coefficients c^λ are signed integers and the partitions (λ) are all of weight

$$n=N(N-1) \tag{1-4}$$

Di Francesco *et al*[1], following upon earlier work by Dunne[2] have discussed properties of the partitions arising in Eq(1-3). For a given N the partitions are bounded by a highest partition $(2N-2,2N-4,\ldots,0)$ and a lowest partition $((N-1)^{N-1})$ with the partitions being of length N and $N-1$.

2. Admissible partitions

Let

$$n_k=\sum_{i=0}^{k}\lambda_{N-i}-k(k+1)\quad k=0,1,\ldots,N-1 \tag{2-1}$$

Then Di Francesco *et al* define *admissible partitions* as satisfying Eq(2-1) with *all* $n_k\geq 0$. Thus for $N=4$ (642) is an admissible partition whereas (651) is not. They were able to compute the number of admissible partitions A_N for a given N and gave a table for $N\leq 29$. They conjectured that A_N was the number of distinct partitions arising in the expansion, Eq(1-3), provided none of the coefficients vanished. Scharf *et al*[3] showed by explicit calculation that the conjecture failed at $N=8$ as the coefficients vanished for eight of the admissible partitions. For $N=9$ the coefficients vanished for sixty six of the admissible partitions while for $N=10$ the coefficients vanished for 389 of theadmissible partitions. The relevant partitions are listed in Table 1.

Remark 1. Is it possible to give a general result for the number of admissible partitions with vanishing coefficients or better still to determine for which of the admissible partitions the coefficients vanish?

Table 1. Admissible partitions whose coefficients are zero

$N = 8$

$\{13\ 11\ 985^{2}41\}$	$\{13\ 11\ 9854^{2}2\}$	$\{13\ 11\ 976541\}$	$\{13\ 10\ 9^{2}6531\}$
$\{13\ 10\ 987531\}$	$\{12\ 11\ 97^{2}4^{2}2\}$	$\{12\ \ 10^{2}96531\}$	$\{12\ \ 10^{2}7^{2}532\}$

$N = 9$

$\{16\ 13\ 11\ 985^{2}41\}$	$\{16\ 13\ 11\ 9854^{2}2\}$	$\{16\ 13\ 11\ 976541\}$	$\{16\ 13\ 10\ 9^{2}6531\}$
$\{16\ 13\ 10\ 987531\}$	$\{16\ 12\ 11\ 97^{2}4^{2}2\}$	$\{16\ 12\ \ 10^{2}96531\}$	$\{16\ 12\ \ 10^{2}7^{2}532\}$
$\{15\ 14\ 11\ 985^{2}41\}$	$\{15\ 14\ 11\ 9854^{2}2\}$	$\{15\ 14\ 11\ 976541\}$	$\{15\ 14\ 10\ 9^{2}6531\}$
$\{15\ 14\ 10\ 987531\}$	$\{15\ 13\ 11\ 10\ 7^{2}63\}$	$\{15\ 13\ 11\ 10\ 7^{2}621\}$	$\{15\ 13\ 11\ 10\ 7^{2}52^{2}\}$
$\{15\ 13\ 11\ 10\ 76^{2}4\}$	$\{15\ 13\ 11\ 9^{2}652^{2}\}$	$\{15\ 13\ 11\ 9^{2}64^{2}1\}$	$\{15\ 13\ 11\ 9^{2}6432\}$
$\{15\ 13\ 11\ 98763\}$	$\{15\ 13\ 11\ 987621\}$	$\{15\ 13\ 11\ 976^{2}32\}$	$\{15\ 13\ 11\ 976542\}$
$\{15\ 13\ \ 10^{2}85^{2}42\}$	$\{15\ 13\ 9^{2}7^{2}543\}$	$\{15\ \ 12^{2}10\ 7^{2}531\}$	$\{15\ \ 12^{2}9^{2}5^{2}41\}$
$\{15\ \ 12^{2}9^{2}54^{2}2\}$	$\{15\ 12\ \ 11^{2}8753\}$	$\{15\ 12\ \ 11^{2}87521\}$	$\{15\ 12\ \ 11^{2}7^{2}4^{2}1\}$
$\{15\ 12\ \ 11^{2}7^{2}432\}$	$\{15\ 12\ 11\ 10\ 9753\}$	$\{15\ 12\ 11\ 10\ 97521\}$	$\{15\ 12\ 11\ 10\ 76^{2}41\}$
$\{15\ 12\ \ 10^{2}96541\}$	$\{15\ 10\ 9^{3}5^{4}\}$	$\{14^{2}11\ 10\ 7^{2}531\}$	$\{14^{2}11\ 9^{2}6531\}$
$\{14\ 13\ 12\ 10\ 7^{2}531\}$	$\{14\ 13\ 12\ 9^{2}5^{2}41\}$	$\{14\ 13\ 12\ 9^{2}54^{2}2\}$	$\{14\ 13\ 11\ 9^{2}6^{2}4\}$
$\{14\ 13\ \ 10^{2}97531\}$	$\{14\ \ 12^{2}11\ 8753\}$	$\{14\ \ 12^{2}11\ 87521\}$	$\{14\ \ 12^{2}11\ 7^{2}4^{2}1\}$
$\{14\ \ 12^{2}11\ 7^{2}432\}$	$\{14\ \ 12^{2}9^{2}754\}$	$\{14\ 12\ \ 11^{2}86^{2}31\}$	$\{14\ 12\ 11\ 10\ 97531\}$
$\{13^{3}10\ 7643^{2}\}$	$\{13^{2}12\ 10\ 963^{3}\}$	$\{13\ 12\ 11\ 9^{2}7^{2}31\}$	$\{13\ 12\ \ 10^{2}95^{3}3\}$
$\{13\ \ 11^{3}76^{2}43\}$	$\{12^{2}987^{3}64\}$	$\{12\ \ 10^{3}76^{3}5\}$	$\{12\ 10\ 9^{3}874^{2}\}$
$\{12\ 10\ 9^{3}85^{3}\}$	$\{11^{4}7^{3}61\}$	$\{11^{3}976^{3}5\}$	$\{11^{3}87^{3}64\}$
$\{11\ \ 10^{3}975^{3}\}$	$\{11\ \ 10^{3}96^{3}4\}$		

$N = 10$

$\{18\ 16\ 13\ 11\ 985^{2}41\}$	$\{18\ 16\ 13\ 11\ 9854^{2}2\}$	$\{18\ 16\ 13\ 11\ 976541\}$	$\{18\ 16\ 13\ 10\ 9^{2}6531\}$
$\{18\ 16\ 13\ 10\ 987531\}$	$\{18\ 16\ 12\ 11\ 97^{2}4^{2}2\}$	$\{18\ 16\ 12\ \ 10^{2}96531\}$	$\{18\ 16\ 12\ \ 10^{2}7^{2}532\}$
$\{18\ 15\ 14\ 11\ 985^{2}41\}$	$\{18\ 15\ 14\ 11\ 9854^{2}2\}$	$\{18\ 15\ 14\ 11\ 976541\}$	$\{18\ 15\ 14\ 10\ 9^{2}6531\}$
$\{18\ 15\ 14\ 10\ 987531\}$	$\{18\ 15\ 13\ 11\ 10\ 7^{2}63\}$	$\{18\ 15\ 13\ 11\ 10\ 7^{2}621\}$	$\{18\ 15\ 13\ 11\ 10\ 7^{2}52^{2}\}$
$\{18\ 15\ 13\ 11\ 10\ 76^{2}4\}$	$\{18\ 15\ 13\ 11\ 9^{2}652^{2}\}$	$\{18\ 15\ 13\ 11\ 9^{2}64^{2}1\}$	$\{18\ 15\ 13\ 11\ 9^{2}6432\}$
$\{18\ 15\ 13\ 11\ 98763\}$	$\{18\ 15\ 13\ 11\ 987621\}$	$\{18\ 15\ 13\ 11\ 976^{2}32\}$	$\{18\ 15\ 13\ 11\ 976542\}$
$\{18\ 15\ 13\ \ 10^{2}85^{2}42\}$	$\{18\ 15\ 13\ 9^{2}7^{2}543\}$	$\{18\ 15\ \ 12^{2}10\ 7^{2}531\}$	$\{18\ 15\ \ 12^{2}9^{2}5^{2}41\}$
$\{18\ 15\ \ 12^{2}9^{2}54^{2}2\}$	$\{18\ 15\ 12\ \ 11^{2}8753\}$	$\{18\ 15\ 12\ \ 11^{2}87521\}$	$\{18\ 15\ 12\ \ 11^{2}7^{2}4^{2}1\}$
$\{18\ 15\ 12\ \ 11^{2}7^{2}432\}$	$\{18\ 15\ 12\ 11\ 10\ 9753\}$	$\{18\ 15\ 12\ 11\ 10\ 97521\}$	$\{18\ 15\ 12\ 11\ 10\ 76^{2}41\}$
$\{18\ 15\ 12\ \ 10^{2}96541\}$	$\{18\ 15\ 10\ 9^{3}5^{4}\}$	$\{18\ \ 14^{2}11\ 10\ 7^{2}531\}$	$\{18\ \ 14^{2}11\ 9^{2}6531\}$
$\{18\ 14\ 13\ 12\ 10\ 7^{2}531\}$	$\{18\ 14\ 13\ 12\ 9^{2}5^{2}41\}$	$\{18\ 14\ 13\ 12\ 9^{2}54^{2}2\}$	$\{18\ 14\ 13\ 11\ 9^{2}6^{2}4\}$
$\{18\ 14\ 13\ \ 10^{2}97531\}$	$\{18\ 14\ \ 12^{2}11\ 8753\}$	$\{18\ 14\ \ 12^{2}11\ 87521\}$	$\{18\ 14\ \ 12^{2}11\ 7^{2}4^{2}1\}$
$\{18\ 14\ \ 12^{2}11\ 7^{2}432\}$	$\{18\ 14\ \ 12^{2}9^{2}754\}$	$\{18\ 14\ 12\ \ 11^{2}86^{2}31\}$	$\{18\ 14\ 12\ 11\ 10\ 97531\}$
$\{18\ \ 13^{3}10\ 7643^{2}\}$	$\{18\ \ 13^{2}12\ 10\ 963^{3}\}$	$\{18\ 13\ 12\ 11\ 9^{2}7^{2}31\}$	$\{18\ 13\ 12\ \ 10^{2}95^{3}3\}$
$\{18\ 13\ \ 11^{3}76^{2}43\}$	$\{18\ \ 12^{2}987^{3}64\}$	$\{18\ 12\ \ 10^{3}76^{3}5\}$	$\{18\ 12\ 10\ 9^{3}874^{2}\}$
$\{18\ 12\ 10\ 9^{3}85^{3}\}$	$\{18\ \ 11^{4}7^{3}61\}$	$\{18\ \ 11^{3}976^{3}5\}$	$\{18\ \ 11^{3}87^{3}64\}$

$N = 10(contd)$

$\{18\ 11\ \ 10^3975^3\}$	$\{18\ 11\ \ 10^396^34\}$	$\{17^213\ 11\ 985^241\}$	$\{17^213\ 11\ 9854^22\}$
$\{17^213\ 11\ 976541\}$	$\{17^213\ 10\ 9^26531\}$	$\{17^213\ 10\ 987531\}$	$\{17^212\ 11\ 97^24^22\}$
$\{17^212\ \ 10^296531\}$	$\{17^212\ \ 10^27^2532\}$	$\{17\ 16\ 14\ 11\ 985^241\}$	$\{17\ 16\ 14\ 11\ 9854^22\}$
$\{17\ 16\ 14\ 11\ 976541\}$	$\{17\ 16\ 14\ 10\ 9^26531\}$	$\{17\ 16\ 14\ 10\ 987531\}$	$\{17\ 16\ 13\ 11\ 10\ 7^263\}$
$\{17\ 16\ 13\ 11\ 10\ 7^2621\}$	$\{17\ 16\ 13\ 11\ 10\ 7^252^2\}$	$\{17\ 16\ 13\ 11\ 10\ 76^24\}$	$\{17\ 16\ 13\ 11\ 9^2652^2\}$
$\{17\ 16\ 13\ 11\ 9^264^21\}$	$\{17\ 16\ 13\ 11\ 9^26432\}$	$\{17\ 16\ 13\ 11\ 98763\}$	$\{17\ 16\ 13\ 11\ 987621\}$
$\{17\ 16\ 13\ 11\ 976^232\}$	$\{17\ 16\ 13\ 11\ 976542\}$	$\{17\ 16\ 13\ \ 10^285^242\}$	$\{17\ 16\ 13\ 9^27^2543\}$
$\{17\ 16\ \ 12^210\ 7^2531\}$	$\{17\ 16\ \ 12^29^25^241\}$	$\{17\ 16\ \ 12^29^254^22\}$	$\{17\ 16\ 12\ \ 11^28753\}$
$\{17\ 16\ 12\ \ 11^287521\}$	$\{17\ 16\ 12\ \ 11^27^24^21\}$	$\{17\ 16\ 12\ \ 11^27^2432\}$	$\{17\ 16\ 12\ 11\ 10\ 9753\}$
$\{17\ 16\ 12\ 11\ 10\ 97521\}$	$\{17\ 16\ 12\ 11\ 10\ 76^241\}$	$\{17\ 16\ 12\ \ 10^296541\}$	$\{17\ 16\ 10\ 9^35^4\}$
$\{17\ \ 15^211\ 985^241\}$	$\{17\ \ 15^211\ 9854^22\}$	$\{17\ \ 15^211\ 976541\}$	$\{17\ \ 15^210\ 9^26531\}$
$\{17\ \ 15^210\ 987531\}$	$\{17\ 15\ 14\ \ 11^2763^3\}$	$\{17\ 15\ 14\ 11\ 7^3642\}$	$\{17\ 15\ 14\ 11\ 7^26^243\}$
$\{17\ 15\ 14\ 8^37652\}$	$\{17\ 15\ 14\ 8^27^343\}$	$\{17\ 15\ 13\ 12\ 11\ 763^3\}$	$\{17\ 15\ 13\ 12\ 9^2852\}$
$\{17\ 15\ 13\ 12\ 9^2851^2\}$	$\{17\ 15\ 13\ 12\ 9^2843\}$	$\{17\ 15\ 13\ 12\ 9^28421\}$	$\{17\ 15\ 13\ 12\ 9^283^21\}$
$\{17\ 15\ 13\ 12\ 9^2832^2\}$	$\{17\ 15\ 13\ 12\ 9^274^2\}$	$\{17\ 15\ 13\ 12\ 9^273^22\}$	$\{17\ 15\ 13\ 12\ 9^25^241\}$
$\{17\ 15\ 13\ 12\ 9^254^22\}$	$\{17\ 15\ 13\ 12\ 98^262\}$	$\{17\ 15\ 13\ 12\ 98^261^2\}$	$\{17\ 15\ 13\ 12\ 8^2743^2\}$
$\{17\ 15\ 13\ 12\ 876543\}$	$\{17\ 15\ 13\ \ 11^2874^2\}$	$\{17\ 15\ 13\ \ 11^2873^22\}$	$\{17\ 15\ 13\ \ 11^286^23\}$
$\{17\ 15\ 13\ \ 11^286^221\}$	$\{17\ 15\ 13\ \ 11^28654\}$	$\{17\ 15\ 13\ 11\ 10\ 9852\}$	$\{17\ 15\ 13\ 11\ 10\ 9851^2\}$
$\{17\ 15\ 13\ 11\ 10\ 9843\}$	$\{17\ 15\ 13\ 11\ 10\ 98421\}$	$\{17\ 15\ 13\ 11\ 10\ 983^21\}$	$\{17\ 15\ 13\ 11\ 10\ 9832^2\}$
$\{17\ 15\ 13\ 11\ 10\ 75^32\}$	$\{17\ 15\ 13\ 11\ 10\ 75^243\}$	$\{17\ 15\ 13\ 11\ 98^254\}$	$\{17\ 15\ 13\ 11\ 98764\}$
$\{17\ 15\ 13\ 11\ 8^362^2\}$	$\{17\ 15\ 13\ 11\ 876^243\}$	$\{17\ 15\ 13\ \ 10^295^31\}$	$\{17\ 15\ 13\ \ 10^2954^23\}$
$\{17\ 15\ 13\ 10\ 9^25^32\}$	$\{17\ 15\ 13\ 10\ 9^25^243\}$	$\{17\ 15\ 13\ 10\ 97^2642\}$	$\{17\ 15\ 13\ 9^37632\}$
$\{17\ 15\ 13\ 9^2875^22\}$	$\{17\ 15\ \ 12^37652^2\}$	$\{17\ 15\ \ 12^3764^21\}$	$\{17\ 15\ \ 12^376432\}$
$\{17\ 15\ \ 12^375^232\}$	$\{17\ 15\ \ 12^210\ 7^264\}$	$\{17\ 15\ 12\ \ 11^275^32\}$	$\{17\ 15\ 12\ \ 11^275^243\}$
$\{17\ 15\ 12\ 11\ 97^2543\}$	$\{17\ 15\ 12\ \ 10^28^2631\}$	$\{17\ 15\ \ 11^29^2765\}$	$\{17\ \ 14^213\ 87^243^2\}$
$\{17\ \ 14^212\ 11\ 6^331\}$	$\{17\ \ 14^212\ 11\ 6^2541\}$	$\{17\ \ 14^212\ 11\ 65^23^2\}$	$\{17\ \ 14^212\ 9^2753\}$
$\{17\ \ 14^212\ 9^27521\}$	$\{17\ \ 14^212\ 9^25^241\}$	$\{17\ \ 14^212\ 9^254^22\}$	$\{17\ \ 14^2\ 11^27^263\}$
$\{17\ \ 14^2\ 11^27^2621\}$	$\{17\ \ 14^2\ 11^276^24\}$	$\{17\ \ 14^210\ 9^27541\}$	$\{17\ 14\ \ 13^210\ 9752\}$
$\{17\ 14\ \ 13^210\ 9751^2\}$	$\{17\ 14\ \ 13^210\ 9743\}$	$\{17\ 14\ \ 13^210\ 97421\}$	$\{17\ 14\ \ 13^210\ 973^21\}$
$\{17\ 14\ \ 13^210\ 9732^2\}$	$\{17\ 14\ \ 13^29^26^23\}$	$\{17\ 14\ \ 13^29^26^221\}$	$\{17\ 14\ \ 13^29^2654\}$
$\{17\ 14\ \ 13^29^26531\}$	$\{17\ 14\ \ 13^29^2652^2\}$	$\{17\ 14\ \ 13^29^264^21\}$	$\{17\ 14\ \ 13^29^26432\}$
$\{17\ 14\ \ 13^29^263^3\}$	$\{17\ 14\ \ 13^28^27541\}$	$\{17\ 14\ 13\ \ 12^27652^2\}$	$\{17\ 14\ 13\ \ 12^2764^21\}$
$\{17\ 14\ 13\ \ 12^276432\}$	$\{17\ 14\ 13\ \ 12^275^232\}$	$\{17\ 14\ 13\ 12\ 11\ 9752\}$	$\{17\ 14\ 13\ 12\ 11\ 9751^2\}$
$\{17\ 14\ 13\ 12\ 11\ 9743\}$	$\{17\ 14\ 13\ 12\ 11\ 97421\}$	$\{17\ 14\ 13\ 12\ 11\ 973^21\}$	$\{17\ 14\ 13\ 12\ 11\ 9732^2\}$
$\{17\ 14\ 13\ 12\ 11\ 76^22^2\}$	$\{17\ 14\ 13\ 12\ 11\ 75^242\}$	$\{17\ 14\ 13\ 12\ 98^263\}$	$\{17\ 14\ 13\ 12\ 98^2621\}$
$\{17\ 14\ 13\ 12\ 8^362^2\}$	$\{17\ 14\ 13\ 12\ 8^26^251\}$	$\{17\ 14\ 13\ 11\ \ 10^25^241\}$	$\{17\ 14\ 13\ 11\ \ 10^254^22\}$
$\{17\ 14\ 13\ 11\ 9^284^21\}$	$\{17\ 14\ 13\ 11\ 9^28432\}$	$\{17\ 14\ 13\ 9^384^23\}$	$\{17\ 14\ \ 12^211\ 8763\}$
$\{17\ 14\ \ 12^211\ 87621\}$	$\{17\ 14\ \ 12^211\ 765^21\}$	$\{17\ 14\ 12\ \ 10^35^32\}$	$\{17\ 14\ 12\ \ 10^35^243\}$
$\{17\ 14\ 12\ 10\ 9876^21\}$	$\{17\ 14\ 11\ \ 10^3843^2\}$	$\{17\ 14\ 11\ \ 10^36^242\}$	$\{17\ 14\ 11\ \ 10^35^33\}$
$\{17\ \ 13^398^2531\}$	$\{17\ \ 13^2\ 12^275^23^2\}$	$\{17\ \ 13^212\ 11\ 76^241\}$	$\{17\ 13\ \ 12^2\ 10^26541\}$

$N = 10(contd)$

$\{17\ 13\ 12\ 11\ 97^{3}43\}$	$\{17\ 12^{2}11\ 10\ 98641\}$	$\{17\ 12\ 11^{3}7^{4}\}$	$\{17\ 12\ 10^{5}4^{2}3\}$
$\{17\ 12\ 10^{4}94^{3}\}$	$\{17\ 12\ 10^{4}65^{3}\}$	$\{17\ 11^{2}10\ 9^{3}83^{2}\}$	$\{16^{2}15\ 11\ 985^{2}41\}$
$\{16^{2}15\ 11\ 9854^{2}2\}$	$\{16^{2}15\ 11\ 976541\}$	$\{16^{2}15\ 10\ 9^{2}6531\}$	$\{16^{2}15\ 10\ 987531\}$
$\{16^{2}13\ 12\ 11\ 6^{3}31\}$	$\{16^{2}13\ 12\ 11\ 6^{2}541\}$	$\{16^{2}13\ 12\ 11\ 65^{2}3^{2}\}$	$\{16^{2}13\ 12\ 9^{2}753\}$
$\{16^{2}13\ 12\ 9^{2}7521\}$	$\{16^{2}13\ 12\ 9^{2}5^{2}41\}$	$\{16^{2}13\ 12\ 9^{2}54^{2}2\}$	$\{16^{2}13\ 11^{2}8753\}$
$\{16^{2}13\ 11^{2}87521\}$	$\{16^{2}\ 12^{2}11\ 76541\}$	$\{16^{2}12\ 10^{3}7531\}$	$\{16^{2}12\ 10^{3}6541\}$
$\{16^{2}11\ 10^{3}5^{3}2\}$	$\{16^{2}11\ 10^{3}5^{2}43\}$	$\{16\ 15^{2}11\ 10\ 7^{2}531\}$	$\{16\ 15^{2}11\ 9^{2}6531\}$
$\{16\ 15\ 14\ 13\ 87^{2}43^{2}\}$	$\{16\ 15\ 14\ 12\ 11\ 6^{3}31\}$	$\{16\ 15\ 14\ 12\ 11\ 6^{2}541\}$	$\{16\ 15\ 14\ 12\ 11\ 65^{2}3^{2}\}$
$\{16\ 15\ 14\ 12\ 9^{2}753\}$	$\{16\ 15\ 14\ 12\ 9^{2}7521\}$	$\{16\ 15\ 14\ 12\ 9^{2}5^{2}41\}$	$\{16\ 15\ 14\ 12\ 9^{2}54^{2}2\}$
$\{16\ 15\ 14\ 11^{2}7^{2}63\}$	$\{16\ 15\ 14\ 11^{2}7^{2}621\}$	$\{16\ 15\ 14\ 11^{2}76^{2}4\}$	$\{16\ 15\ 14\ 10\ 9^{2}7541\}$
$\{16\ 15\ 13^{2}11\ 6^{3}31\}$	$\{16\ 15\ 13^{2}11\ 6^{2}541\}$	$\{16\ 15\ 13^{2}11\ 65^{2}3^{2}\}$	$\{16\ 15\ 13\ 11^{2}8^{2}62\}$
$\{16\ 15\ 13\ 11^{2}8^{2}61^{2}\}$	$\{16\ 15\ 12^{2}11\ 9753\}$	$\{16\ 15\ 12^{2}11\ 97521\}$	$\{16\ 15\ 12\ 11\ 9^{3}531\}$
$\{16\ 15\ 12\ 10^{2}8763^{2}\}$	$\{16\ 14^{2}13\ 10\ 9752\}$	$\{16\ 14^{2}13\ 10\ 9751^{2}\}$	$\{16\ 14^{2}13\ 10\ 9743\}$
$\{16\ 14^{2}13\ 10\ 97421\}$	$\{16\ 14^{2}13\ 10\ 973^{2}1\}$	$\{16\ 14^{2}13\ 10\ 9732^{2}\}$	$\{16\ 14^{2}13\ 9^{2}6^{2}3\}$
$\{16\ 14^{2}13\ 9^{2}6^{2}21\}$	$\{16\ 14^{2}13\ 9^{2}654\}$	$\{16\ 14^{2}13\ 9^{2}6531\}$	$\{16\ 14^{2}13\ 9^{2}652^{2}\}$
$\{16\ 14^{2}13\ 9^{2}64^{2}1\}$	$\{16\ 14^{2}13\ 9^{2}6432\}$	$\{16\ 14^{2}13\ 9^{2}63^{3}\}$	$\{16\ 14^{2}13\ 8^{2}7541\}$
$\{16\ 14^{2}\ 11^{2}9762\}$	$\{16\ 14^{2}\ 11^{2}9761^{2}\}$	$\{16\ 14^{2}11\ 987542\}$	$\{16\ 14\ 13^{2}11\ 76541\}$
$\{16\ 14\ 13^{2}10\ 8^{2}53\}$	$\{16\ 14\ 13^{2}10\ 8^{2}521\}$	$\{16\ 14\ 13\ 12\ 11\ 9753\}$	$\{16\ 14\ 13\ 12\ 11\ 97521\}$
$\{16\ 14\ 13\ 11\ 10\ 974^{2}2\}$	$\{16\ 14\ 12^{2}8^{3}741\}$	$\{16\ 14\ 12\ 11^{3}7431\}$	$\{16\ 14\ 12\ 11^{2}98531\}$
$\{16\ 14\ 12\ 11\ 987652\}$	$\{16\ 13^{3}12\ 75^{2}3^{2}\}$	$\{16\ 13^{3}11\ 87531\}$	$\{16\ 13^{3}11\ 7^{2}631\}$
$\{16\ 13^{3}9^{2}8531\}$	$\{16\ 13^{3}8^{3}72^{2}\}$	$\{16\ 13^{3}8^{3}641\}$	$\{16\ 13^{2}11\ 10\ 9^{2}531\}$
$\{16\ 13\ 12^{2}11\ 76^{2}43\}$	$\{16\ 13\ 12\ 11\ 10^{3}431\}$	$\{16\ 13\ 12\ 11\ 10\ 97642\}$	$\{16\ 11^{2}10\ 987^{3}4\}$
$\{16\ 11^{2}987^{5}\}$	$\{16\ 11\ 10^{5}54^{2}\}$	$\{15^{3}12\ 11\ 7^{2}431\}$	$\{15^{3}12\ 11\ 76531\}$
$\{15^{3}12\ 9^{2}5^{2}41\}$	$\{15^{3}12\ 9^{2}54^{2}2\}$	$\{15^{3}12\ 9865^{2}\}$	$\{15^{2}14\ 12\ 11\ 85^{3}\}$
$\{15^{2}14\ 12\ 11\ 84^{3}3\}$	$\{15^{2}14\ 11^{2}10\ 54^{2}1\}$	$\{15^{2}14\ 11^{2}10\ 5432\}$	$\{15^{2}14\ 11\ 10^{2}6531\}$
$\{15^{2}14\ 10\ 8^{3}741\}$	$\{15^{2}\ 13^{2}12\ 7652^{2}\}$	$\{15^{2}\ 13^{2}12\ 764^{2}1\}$	$\{15^{2}\ 13^{2}12\ 76432\}$
$\{15^{2}\ 13^{2}12\ 75^{2}32\}$	$\{15^{2}\ 13^{2}11\ 6^{2}5^{2}1\}$	$\{15^{2}\ 13^{2}11\ 65^{3}2\}$	$\{15^{2}\ 13^{2}11\ 65^{2}43\}$
$\{15^{2}12\ 11\ 10\ 8^{2}632\}$	$\{15^{2}10\ 9^{3}87^{2}1\}$	$\{15\ 14^{3}10\ 7643^{2}\}$	$\{15\ 14^{2}13\ 98^{2}531\}$
$\{15\ 14^{2}10\ 9^{3}541\}$	$\{15\ 14^{2}8^{5}61\}$	$\{15\ 14\ 13^{2}12\ 75^{2}3^{2}\}$	$\{15\ 14\ 13^{2}11\ 87531\}$
$\{15\ 14\ 13^{2}11\ 7^{2}631\}$	$\{15\ 14\ 13^{2}9^{2}8531\}$	$\{15\ 14\ 13^{2}8^{3}72^{2}\}$	$\{15\ 14\ 13^{2}8^{3}641\}$
$\{15\ 14\ 13\ 12\ 11\ 10\ 6531\}$	$\{15\ 14\ 13\ 11^{2}9^{2}53\}$	$\{15\ 14\ 13\ 11^{2}9^{2}521\}$	$\{15\ 14\ 13\ 11^{2}97631\}$
$\{15\ 14\ 12^{2}\ 11^{2}7431\}$	$\{15\ 14\ 12^{2}11\ 10\ 7531\}$	$\{15\ 14\ 12^{2}11\ 7^{3}5\}$	$\{15\ 14\ 12^{2}11\ 76^{2}52\}$
$\{15\ 14\ 12^{2}11\ 765^{2}3\}$	$\{15\ 14\ 11^{3}\ 10^{2}431\}$	$\{15\ 14\ 11^{3}97651\}$	$\{15\ 13^{3}98^{2}65\}$
$\{15\ 13^{3}8^{3}741\}$	$\{15\ 13^{2}12\ 11\ 76^{2}43\}$	$\{15\ 12^{2}10\ 8^{2}7^{2}65\}$	$\{15\ 12\ 11\ 9^{2}8^{2}7^{2}4\}$
$\{15\ 11^{2}10\ 9^{2}7^{3}4\}$	$\{15\ 11\ 10^{3}98764\}$	$\{14^{3}98^{4}61\}$	$\{14^{2}13\ 8^{5}72\}$
$\{14^{2}13\ 8^{2}7^{4}5\}$	$\{14^{2}11\ 10\ 9^{3}86\}$	$\{14\ 12^{3}98^{3}7\}$	$\{14\ 12\ 11^{3}10\ 96^{2}\}$

$N = 10 (contd)$

$\{14\ 12\ 11^3 10\ 7^3\}$ $\{14\ 12\ 11\ 10\ 9 8^3 7 3\}$ $\{14\ 11^3 10\ 9 8 7^2 2\}$ $\{14\ 11^3 9^2 8 7^2 3\}$
$\{14\ 11^2\ 10^2 9^2 7 6 3\}$ $\{14\ 11^2 10\ 9^2 7^3 5\}$ $\{14\ 11\ 10^2 8^3 7^3\}$ $\{13^4 9^3 8 3\}$
$\{13^4 9^3 8 2 1\}$ $\{13^3 12\ 8^4 6 1\}$ $\{13^3 11\ 9 8^3 7\}$ $\{13^3 10\ 9^3 8 6\}$
$\{13\ 12^3 11\ 9 7^3\}$ $\{13\ 12^3 11\ 8^3 6\}$ $\{13\ 12\ 11^2\ 10^2 8 6^2 3\}$ $\{13\ 12\ 11\ 8^5 7^2\}$
$\{13\ 11^4\ 10^2 5 4^2\}$ $\{13\ 11^3 9^2 8 7^2 4\}$ $\{11^5 10\ 9 7^2 2\}$ $\{11^3\ 10^3 8^2 7 4\}$
$\{11^2\ 10^5 7 6 5\}$

3. Calculating the terms in the expansion

Scharf *et al* discussed algorithms for calculating the coefficients in the expansion, Eq(3). Let us write

$$U_N(z_1,\ldots,z_N) = \prod_{i=1}^{N-1}(z_i - z_N)^2 \tag{3-1}$$

We can express U_N as a sum of monomials $z_1^{m_1} z_2^{m_2} \ldots z_N^{m_n}$ where $m_1+m_2+\ldots+m_n = N$. For brevity we put

$$z_1^{m_1} z_2^{m_2} \ldots z_N^{m_n} = (m_1 m_2 \ldots m_N) \tag{3-2}$$

Thus, for example,

$$\begin{aligned} U_3(z_1,z_2,z_3) &= (z_1^2 z_2^2) + (z_1^2 z_3^2) + (z_2^2 z_3^2) + (z_3^4) - 2(z_1^2 z_2 z_3) - 2(z_1 z_2^2 z_3) \\ &\quad - 2(z_2 z_3^3) - 2(z_1 z_3^3) + 4(z_1 z_2 z_3^2) \\ &= (2^2) + (202) + (02^2) + (0^2 4) - 2(21^2) - 2(013) \\ &\quad - 2(121) - 2(103) + 4(1^2 2) \end{aligned} \tag{3-3}$$

Note that the terms in Eq(3-3) can be grouped with respect to their first part to give

$$\begin{aligned} U_3 = &\{(2^2) + (202) - 2(21^2)\} + \{4(1^2 2) - 2(121) - 2(103)\} \\ &+ \{(02^2) + (0^2 4) - 2(013)\} \end{aligned} \tag{3-4}$$

Quite generally the monomials in the expansion of U_N can all be grouped into three subsets whose first part is $0, 1$ and 2. Thus

$$U_N = U_N^{(0)} + U_N^{(1)} + U_N^{(2)} \tag{3-5}$$

One may construct the expansion in Eq(1-3) recursively as

$$V_N^2 = U_N V_{N-1}^2 = \sum_{\lambda \vdash n} c^\lambda \{\lambda\} \tag{3-6}$$

The Schur functions in the expansion of V_{N-1}^2 can likewise be divided into subsets according to the first part of the partitions indexing the Schur functions to give

$$V_{N-1}^2 = V_{N-1}^2(2N-4) + V_{N-1}^2(2N-5) + \ldots + V_{N-1}^2(N-1) \tag{3-7}$$

In making practical calculations it is convenient to form the expansions of the terms $U_N^{(i)} V_{N-1}^2(j)$ separately into Schur functions and then to combine the results.

Remark 2 When the expansions are carried out separately many of the products involve the combination of many Schur functions which sum to zero. In Table 2 we indicate products summing to zero by a 0 and those summing to a non-vanishing set of Schur functions by a x. These are displayed symbolically as matrices, one for each V_N^2 for $N = 2, \ldots, 8$. The columns of a given matrix is indexed by the numbers j arising in Eq(13) and the rows by the numbers i arising in Eq(11). Clearly it would be advantageous to be able to predict those expansions that give zero contributions.

Table 2. Symbolic matrices based upon Eqs (3-5) and (3-7)

$$\begin{array}{cc} V_2^2 & 0 \\ 0 & x \\ 1 & x \\ 2 & x \end{array} \qquad \begin{array}{ccc} V_3^2 & 2 & 1 \\ 0 & x & 0 \\ 1 & 0 & x \\ 2 & x & x \end{array} \qquad \begin{array}{cccc} V_4^2 & 4 & 3 & 2 \\ 0 & 0 & x & x \\ 1 & 0 & 0 & x \\ 2 & x & x & x \end{array} \qquad \begin{array}{ccccc} V_5^2 & 6 & 5 & 4 & 3 \\ 0 & 0 & 0 & x & x \\ 1 & 0 & 0 & 0 & x \\ 2 & x & x & x & x \end{array}$$

$$\begin{array}{cccccc} V_6^2 & 8 & 7 & 6 & 5 & 4 \\ 0 & 0 & 0 & x & x & x \\ 1 & 0 & 0 & 0 & 0 & x \\ 2 & x & x & x & x & x \end{array} \qquad \begin{array}{ccccccc} V_7^2 & 10 & 9 & 8 & 7 & 6 & 5 \\ 0 & 0 & 0 & 0 & x & x & x \\ 1 & 0 & 0 & 0 & 0 & 0 & x \\ 2 & x & x & x & x & x & x \end{array}$$

$$\begin{array}{cccccccc} V_8^2 & 12 & 11 & 10 & 9 & 8 & 7 & 6 \\ 0 & 0 & 0 & 0 & x & x & x & x \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 2 & x & x & x & x & x & x & x \end{array} \qquad \begin{array}{ccccccccc} V_9^2 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 \\ 0 & 0 & 0 & 0 & 0 & x & x & x & x \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \\ 2 & x & x & x & x & x & x & x & x \end{array}$$

4. The $q-$discriminant

Let $q\mathbf{x} = (qx_1, qx_2, \ldots, qx_N)$ and the $q-$discriminant of $\mathbf{x}$ be

$$D_N(q;\mathbf{x}) = \prod_{1 \le i \ne j \le N} (x_i - qx_j) \tag{4-1}$$

and

$$R_N(q;\mathbf{x}) = \prod_{1 \le i \ne j \le N} (x_i - qx_j)(qx_i - x_j) = \sum_\lambda c^\lambda(q) s_\lambda(\mathbf{x}) \tag{4-2}$$

So that

$$V_N^2(\mathbf{x}) = \prod_{1\le i<j\le N} (x_i - x_j)^2 = R_N(1;\mathbf{x}) \tag{4-3}$$

Introduce $q-$polynomials such that

$$R_N(q;\mathbf{x}) = \sum_\lambda c^\lambda(q) s_\lambda(\mathbf{x}) \tag{4-4}$$

$$R_N(q;\mathbf{x}) = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu\subseteq(N-1)^N} ((-q)^{|\nu|)} + (-q)^{N^2-|\nu|}) \times s_{(N-1)^N/\nu}(\mathbf{x}) s_{\nu'}(\mathbf{x}) \tag{4-5}$$

5. $q-$Expansion of the square of the Vandermonde determinant - factored form

The first entry in square brackets is the value of the $q-$polynomial for $q = 1$

N = 2

$0, 1, [1] \quad q\{2\}$
$1, 0, [-3] - (q^2 + q + 1)\{1^2\}$

N = 3

$2, 7, [1] \quad q^3\{42\}$
$3, 6, [-3] \quad - q^2(q^2 + q + 1)(\{41^2\} + \{3^2\})$
$4, 4, [6] \quad + q(q^2 + q + 1)(q^2 + 1)\{321\}$
$6, 3, [-15] - (q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)\{2^3\}$

N = 4

$8, 22, [1] \quad q^6\{642\}$
$9, 21, [-3] \quad - q^5(q^2 + q + 1)(\{641^2\} + \{63^2\} + \{5^22\})$
$10, 19, [6] \quad + q^4(q^2 + 1)(q^2 + q + 1)(\{6321\} + \{543\})$
$10, 20, [9] \quad + q^4(q^2 + q + 1)^2\{5^21^2\}$
$12, 18, [-15] - q^3(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(\{62^3\} + \{4^3\})$
$11, 17, [-12] - q^3(q^2 + q + 1)(q^2 + 1)^2\{5421\}$
$12, 16, [-9] \quad - q^3(q^4 + q^2 + 1)(q^2 + q + 1)\{53^21\}$
$22, 15, [-6] \quad - q^3(q^2 + q + 1)(q^4 + 1)\{4^22^2\}$
$13, 15, [27] \quad + q^2(q^4 + q^2 + 1)(q^2 + q + 1)^2(\{532^2\} + \{4^231\})$
$15, 13, [-45] - q(q^4 + q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)\{43^22\}$
$18, 12, [105] \quad + (q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)\{3^4\}$

N = 5

$20, 50, [1]\quad q^{10}\{8642\}$

$21, 49, [-3]\quad -q^9(q^2+q+1)(\{8641^2\}+\{863^2\}+\{85^22\}+\{7^242\})$

$22, 47, [6]\quad +q^8(q^2+1)(q^2+q+1)(\{86321\}+\{8543\}+\{7652\})$

$22, 48, [9]\quad +q^8(q^2+q+1)^2(\{85^21^2\}+\{7^241^2\}+\{7^23^2\})$

$23, 45, [-12]\quad -q^7(q^2+q+1)(q^2+1)^2(\{85421\}+\{7643\})$

$24, 44, [-9]\quad -q^7(q^4+q^2+1)(q^2+q+1)(\{853^21\}+\{75^23\})$

$26, 42, [-6]\quad -q^7(q^2+q+1)(q^4+1)(\{84^22^2\}+\{6^24^2\}$

$24, 46, [-15]\quad -q^7(q^2+q+1)(q^4+q^3+q^2+q+1)(\{862^3\}+\{84^3\}+\{6^32\})$

$23, 46, [-18]\quad -q^7(q^2+1)(q^2+q+1)^2(\{7^2321\}+\{7651^2\})$

$28, 38, [-18]\quad -q^5(q^4+1)(q^2+q+1)(q^4+q^2+1)(\{6^23^22\}+\{65^22^2\})$

$25, 43, [27]\quad +q^6(q^4+q^2+1)(q^2+q+1)^2(\{8532^2\}+\{84^231\}+\{754^2\}+\{6^253\})$

$20, 43, [24]\quad +q^6(q^2+q+1)(q^2+1)^3\{76421\}$

$25, 42, [18]\quad +q^6(q^2+1)(q^2+q+1)(q^4+q^2+1)(\{763^21\}+\{75^221\})$

$25, 45, [45]\quad +q^6(q^4+q^3+q^2+q+1)(q^2+q+1)^2(\{7^22^3\}+\{6^31^2\})$

$27, 41, [-45]\quad -q^5(q^4+q^3+q^2+q+1)(q^2+q+1)(q^4+q^2+1)(\{843^22\}+\{65^24\})$

$26, 41, [-54]\quad -q^5(q^2+1)(q^2+q+1)^2(q^4+q^2+1)(\{7632^2\}+\{6^2521\})$

$26, 40, [-36]\quad -q^5(q^2+1)^2(q^2-q+1)(q^2+q+1)^2\{75431\}$

$28, 39, [-36]\quad -q^5(q^2+1)(q^2+q+1)^2(q^4+1)\{74^31\}$

$27, 39, [-27]\quad -q^5(q^2+q+1)(q^4+q^2+1)^2(\{7542^2\}+\{6^2431\})$

$30, 40, [105]\quad +q^4(q^2+q+1)(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$
$(\{83^4\}+\{5^4\})$

$28, 38, [81]\quad +q^4(q^2+q+1)^2(q^4+q^2+1)^2(\{753^22\}+\{65^231\})$

$29, 37, [72]\quad +q^4(q^4+1)(q^2+1)^2(q^2+q+1)^2(\{74^232\}+\{654^21\})$

$28, 38, [111]\quad +q^4(q^2+q+1)$
$\times(q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4+3q^3+4q^2+2q+1)\{6^242^2\}$

$32, 34, [45]\quad +q^4(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)$
$\times(q^2+q+1)^2(\{653^3\}+\{5^332\})$

$31, 36, [-180]\quad -q^3(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)(q^2+q+1)^2(\{743^3\}+\{5^341\})$

$30, 35, [-144]\quad -q^3(q^4+1)(q^2+q+1)^2(q^2+1)^3\{65432\}$

$32, 34, [-90]\quad -q^3(q^4+q^3+q^2+q+1)(q^4+1)(q^2+q+1)(q^4+q^2+1)\{64^32\}$

$34, 32, [-75]\quad -q^3(q^2+q+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2\{5^243^2\}$

$33, 33[270]\quad +q^2(q^4+q^3+q^2+q+1)(q^2-q+1)(q^4+1)(q^2+q+1)^3$
$(\{64^23^2\}+\{5^24^22\})$

$36, 31, [-420]\quad -q(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)$
$(q^6+q^5+q^4+q^3+q^2+q+1)\{54^33\}$

40, 30, [945] $+ (q^4 + q^3 + q^2 + q + 1)(q^6 + q^3 + 1)(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)^2\{4^5\}$

N = 6

45, 95, [1] $q^{15}\{10\ 8642\}$

41, 94, [−3] $- q^{14}(q^2 + q + 1)(\{10\ 8641^2\} + \{10\ 863^2\} + \{10\ 85^22\} + \{10\ 7^242\} + \{9^2642\})$

42, 92, [6] $+ q^{13}(q^2 + q + 1)(q^2 + 1)(\{10\ 86321\} + \{10\ 8543\} + \{10\ 7652\} + \{98742\})$

42, 93, [9] $+ q^{13}(q^2 + q + 1)^2(\{10\ 85^21^2\} + \{10\ 7^241^2\} + \{10\ 7^23^2\} + \{9^2641^2\} + \{9^263^2\} + \{9^25^22\})$

44, 91, [−15] $- q^{12}(q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)(\{10\ 862^3\} + \{10\ 84^3\} + \{10\ 6^32\} + \{8^342\})$

44, 89, [−9] $- q^{12}(q^2 + q + 1)(q^4 + q^2 + 1)(\{10\ 853^21\} + \{10\ 75^23\} + \{97^252\})$

43, 90, [−12] $- q^{12}(q^2 + q + 1)(q^2 + 1)^2(\{10\ 85421\} + \{10\ 7643\} + \{98652\})$

46, 87, [−6] $- q^{12}(q^2 + q + 1)(q^4 + 1)(\{10\ 84^22^2\} + \{10\ 6^24^2\} + \{8^26^22\})$

43, 91, [−18] $- q^{12}(q^2 + 1)(q^2 + q + 1)^2(\{10\ 7^2321\} + \{10\ 7651^2\} + \{9^26321\} + (\{9^2543\} + \{98741^2\} + \{9873^2\})$

49, 82[−18] $- q^{10}(q^4 + 1)(q^2 + q + 1)(q^4 + q^2 + 1)(\{10\ 6^23^22\} + \{10\ 65^22^2\} + \{8^25^24\} + \{87^24^2\})$

43, 92, [−27] $- q^{12}(q^2 + q + 1)^3\{9^25^21^2\}$

47, 84, [−27] $- q^{10}(q^2 + q + 1)(q^4 + q^2 + 1)^2(\{10\ 7542^2\} + \{10\ 6^2431\} + \{97^23^21\} + \{9764^2\} + \{8^2653\})$

45, 88, [27] $+ q^{11}(q^2 + q + 1)^2(q^4 + q^2 + 1)(\{10\ 8532^2\} + \{10\ 84^231\} + \{10\ 754^2\} + \{10\ 6^253\} + \{9^253^21\} + \{97^251^2\} + \{976^22\} + \{8^2752\})$

44, 88, [24] $+ q^{11}(q^2 + q + 1)(q^2 + 1)^3(\{10\ 76421\} + \{98643\})$

45, 90, [45] $+ q^{11}(q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)^2(\{10\ 7^22^3\} + \{10\ 6^31^2\} + \{9^262^3\} + \{9^24^3\} + \{8^341^2\} + \{8^33^2\})$

52, 79, [45] $+ q^9(q^4 + q^3 + q^2 + q + 1)(q^4 - q^3 + q^2 - q + 1)(q^2 + q + 1)^2(\{10\ 653^3\} + \{10\ 5^332\} + \{875^3\} + \{7^354\})$

63, 66, [45] $+ q^7(q^2 + q + 1)^2(q^4 - q^2 + 1)(q^4 + q^3 + q^2 + q + 1)(q^4 - q^3 + q^2 - q + 1)(\{7^24^4\} + \{6^43^2\})$

45, 90, [18] $+ q^{11}(q^2 + 1)(q^2 + q + 1)(q^4 + q^2 + 1)(\{10\ 763^21\} + \{10\ 75^221\} + \{985^23\} + \{97^243\})$

47, 86, [18] $+ q^{11}(q^4 + 1)(q^2 + q + 1)^2(\{9^24^22^2\} + \{8^26^21^2\})$

44, 89, [36] $+ q^{11}(q^2 + 1)^2(q^2 + q + 1)^2(\{9^25421\} + \{987321\}) + \{98651^2\})$

$48,84,[-36]\ \ -q^{10}(q^2+1)(q^4+1)(q^2+q+1)^2(\{10\ 74^31\}+\{96^33\})$

$46,85,[-36]\ \ -q^{10}(q^2+1)^2(q^2+q+1)(q^4+q^2+1)(\{10\ 75431\}+\{9863^21\}$
$+\{985^221\}+\{97^2421\}+\{97653\})$

$54,75,[-36]\ \ -q^8(q^2-q+1)(q^2+q+1)^2(q^4+1)^2(\{8^24^32\}+\{86^32^2\})$

$47,86,[-45]\ \ -q^{10}(q^4+q^3+q^2+q+1)(q^2+q+1)(q^4+q^2+1)(\{10\ 843^22\}$
$+\{10\ 65^24\}+\{87^262\})$

$46,86,[-54]\ \ -q^{10}(q^2+1)(q^2+q+1)^2(q^4+q^2+1)(\{10\ 7632^2\}+\{10\ 6^2521\}$
$+\{9854^2\}+\{8^2743\})$

$46,87,[-81]\ \ -q^{10}(q^2+q+1)^3(q^4+q^2+1)(\{9^2532^2\}+\{9^24^231\}+\{976^21^2\}$
$+\{8^2751^2\})$

$46,88,[-90]\ \ -q^{10}(q^2+1)(q^4+q^3+q^2+q+1)(q^2+q+1)^2(\{9872^3\}+\{8^3321\})$

$52,79,[-90]\ \ -q^8(q^4+q^3+q^2+q+1)(q^2-q+1)(q^4+1)(q^2+q+1)^2$
$(\{10\ 64^32\}+\{86^34\})$

$57,72[-90]\ \ -q^8(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^6+q^4-2q^3+q^2+1)\{7^33^3\}$

$45,86,[-48]\ \ -q^{10}(q^2+q+1)(q^2+1)^4\{986421\}$

$50,85,[105]\ \ +q^9(q^2+q+1)(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$
$(\{10\ 83^4\}+\{10\ 5^4\}+\{7^42\})$

$49,83,[81]\ \ +q^9(q^2+q+1)^2(q^4+q^2+1)^2(\{10\ 753^22\}+\{10\ 65^231\}+\{97^232^2\}$
$+\{975^24\}+\{8^273^21\}+\{87^253\})$

$50,79,[81]\ \ +q^9(q^2+q+1)^3(q^4+q^2+1)(2q^2-3q+2)(\{975^22^2\}+\{8^25^231\})$

$49,82,[72]\ \ +q^9(q^4+1)(q^2+1)^2(q^2+q+1)^2(\{10\ 74^232\}+\{10\ 654^21\}+\{984^31\}$
$+\{96^321\}+\{96^254\}+\{876^23\})$

$47,83,[72]\ \ +q^9(q^2+q+1)(q(q^4+q^2+1)^2+1)^3(\{985431\}+\{976521\})$

$48,83,[111]\ \ +q^9(q^2+q+1)(q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4+3q^3$
$+4q^2+2q+1)(\{10\ 6^242^2\}+\{8^264^2\})$

$48,85,[135]\ \ +q^9(q^4+q^3+q^2+q+1)(q^2+q+1)^2(q^4+q^2+1)(\{9^243^22\}$
$+\{87^261^2\})$

$47,84,[108]\ \ +q^9(q^2+1)^2(q^2+q+1)^2(\{98632^2\}+\{8^27421\})$

$53,75,[108]\ \ +q^7(q^2+1)(q^4+1)(q^2-q+1)^2(q^2+q+1)^3(\{965^232\}+\{875^241\})$

$55,74,[108]\ \ +q^7(q^2+q+1)^3(q^2-q+1)(q^4+1)^2(\{8^24^23^2\}+\{7^26^22^2\})$

$48,82,[54]\ \ +q^9(q^2+1)(q^2+q+1)(q^4+q^2+1)^2(\{98542^2\}+\{8^26521\})$

$48,81,[99]\ \ +q^9(q^2+q+1)(q^4+q^2+1)(2q^6+4q^4-q^3+4q^2+2)\{976431\}$

$48,87,[225]\ \ +q^9(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2\{8^32^3\}$

$51,81,[-180]-q^8(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)(q^2+q+1)^2$
$(\{10\ 743^3\}+\{10\ 5^341\}+\{965^3\}+\{7^363\})$

$50, 80, [-144] - q^8(q^4+1)(q^2+q+1)^2(q^2+1)^3$
$(\{10\ 65432\} + \{984^232\} + \{876^221\} + \{87654\})$

$54, 77, [-75]\ \ - q^8(q^2+q+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$
$(\{10\ 5^243^2\} + \{7^265^2\})$

$51, 84, [-315] - q^8(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^2+q+1)^2$
$(\{9^23^4\} + \{7^41^2\})$

$49, 81, [-162] - q^8(q^2+1)(q^2-q+1)^2(q^2+q+1)^4(\{9853^22\} + \{87^2521\})$

$49, 80, [-162] - q^8(q^2+q+1)^3(q^2-q+1)(q^6+3q^4-2q^3+3q^2+1)$
$(\{97642^2\} + \{975^231\} + \{8^26431\})$

$55, 72, [-162] - q^6(q^4+1)(q^2-q+1)^3(q^2+q+1)^4(\{8754^22\} + \{86^2532\})$

$50, 78, [-81]\ \ - q^8(q^2+q+1)^4(q^2-q+1)^3(\{9763^22\} + \{9754^21\} + \{96^2531\}$
$+ \{87^2431\})$

$54, 75, [-81]\ \ - q^8(q^2+q+1)^3(q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$
$(\{96^23^3\} + \{7^34^21\})$

$51, 81, [-117] - q^8(q^2+q+1)^2(q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)$
$(\{96^252^2\} + \{96^24^21\} + \{8^254^21\})$

$49, 82, [-243] - q^8(q^2+q+1)^5(q^2-q+1)^2\{8^2732^2\}$

$50, 79, [-69]\ \ - q^8(q^2+q+1)(q^{12}+2q^{10}+6q^8+5q^6+6q^4+2q^2+1)\{8^2642^2\}$

$51, 78, [-153] - q^8(q^2-q+1)(q^2+q+1)^2$
$(q^8+2q^7+4q^6+q^5+q^4+q^3+4q^2+2q+1)\{8^25^22^2\}$

$52, 77, [-54]\ \ - q^8(q^2-q+1)^2(q^2+q+1)^3(q^4+1)\{87^23^22\}$

$53, 78, [270]\ \ + q^7(q^4+q^3+q^2+q+1)(q^2-q+1)(q^4+1)(q^2+q+1)^3$
$(\{10\ 64^23^2\} + \{10\ 5^24^22\} + \{86^25^2\} + \{7^26^24\})$

$52, 79, [360]\ \ + q^7(q^4+q^3+q^2+q+1)(q^4+1)(q^2+q+1)^2(q^2+1)^2$
$(\{9843^3\} + \{7^3621\})$

$55, 77, [162]\ \ + q^7(q^2+1)(q^2-q+1)^3(q^2+q+1)^4(\{975432\} + \{876531\})$

$53, 76, [162]\ \ + q^7(q^4+1)(q^2-q+1)^2(q^2+q+1)^4(\{9753^3\} + \{974^32\} + \{86^331\}$
$+ \{7^3531\})$

$55, 73, [162]\ \ + q^7(q^2+1)(q^2+q+1)^3(q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$
$(\{8763^3\} + \{7^3432\})$

$52, 76, [234]\ \ + q^7(q^2+1)(q^2+q+1)^2$
$(q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)$
$(\{96^2432\} + \{8^25432\} + \{87652^2\} + \{8764^21\})$

$52, 77, [216]\ \ + q^7(q^2+1)^2(q^4+1)(q^2-q+1)(q^2+q+1)^3\{965^241\}$

$55, 76, [225]\ \ + q^7(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)\{95^41\}$

57, 72, [90] $+ q^7(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^4+1)$
$(\{95^33^2\}+\{7^25^31\})$

51, 78[333] $+ q^7(q^2+q+1)^2(q^2-q+1)$
$\times(q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4+3q^3+4q^2+2q+1)$
$(\{8^263^22\}+\{87^242^2\})$

56, 76, [−420] $- q^6(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)$
$\times(q^6+q^5+q^4+q^3+q^2+q+1)$
$(\{10\ 54^33\}+\{76^35\})$

54, 75, [−486] $- q^6(q^2+q+1)^5(q^2-q+1)^2(q^4+1)(\{974^23^2\}+\{7^26^231\})$

54, 74, [−324] $- q^6(q^2+q+1)^4(q^2-q+1)^2(q^4+1)(q^2+1)(\{9654^22\}+\{86^2541\})$

56, 73, [−405] $- q^6(q^2-q+1)(q^2+q+1)^4(q^4+q^3+q^2+q+1)$
$\times(q^4-q^3+q^2-q+1)(\{95^342\}+\{865^31\})$

54, 75, [−711] $- q^6(q^2+q+1)^2(q^{14}+2q^{13}+4q^{12}+4q^{11}+8q^{10}+7q^9+10q^8$
$+7q^7+10q^6+7q^5+8q^4+4q^3+4q^2+2q+1)$
$(\{8^253^3\}+\{7^352^2\})$

53, 74, [−468] $- q^6(q^2+1)^2(q^2+q+1)^2$
$\times(q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)\{876432\}$

54, 73, [−351] $- q^6(q^2-q+1)(q^2+q+1)^3$
$\times(q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)\{875^232\}$

58, 69, [−201] $- q^6(q^2+q+1)(q^{16}+q^{15}+4q^{14}+2q^{13}+6q^{12}+3q^{11}+9q^{10}$
$+3q^9+9q^8+3q^7+9q^6+3q^5+6q^4+2q^3+4q^2+q+1)\{7^2643^2\}$

57, 70, [−108] $- q^6(q^4+1)^2(q^2-q+1)^2(q^2+q+1)^3(\{86^243^2\}+\{7^264^22\})$

55, 73, [−288] $- q^6(q^4+1)^2(q^2+q+1)^2(q^2+1)^3(\{96543^2\}+\{7^26541\})$

58, 70, [−180] $- q^6(q^2+1)(q^4+1)(q^2+q+1)^2(q^4+q^3+q^2+q+1)$
$\times(q^4-q^3+q^2-q+1)(\{874^33\}+\{76^332\})$

60, 75, [945] $+ q^5(q^4+q^3+q^2+q+1)(q^6+q^3+1)(q^6+q^5+q^4+q^3+q^2+q+1)$
$\times(q^2+q+1)^2(\{10\ 4^5\}+\{6^5\})$

57, 72, [720] $+ q^5(q^4+q^3+q^2+q+1)(q^2+q+1)^2(q^4+1)^2(q^2+1)^2$
$(\{964^33\}+\{76^341\})$

58, 71, [675] $+ q^5(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)(q^2+q+1)^3$
$\times(q^2-q+1)(\{95^24^23\}+\{76^25^21\})$

56, 71, [801] $+ q^5(q^2+q+1)^2(q^{16}+q^{15}+4q^{14}+2q^{13}+9q^{12}+3q^{11}+13q^{10}$
$+4q^9+15q^8+4q^7+13q^6+3q^5+9q^4+2q^3+4q^2+q+1)$
$(\{87543^2\}+\{7^26532\})$

$$
\begin{array}{ll}
56,71,[666] & +q^5(q^4+1)(q^2+q+1)^2(q^2-q+1) \\
& (q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4+3q^3+4q^2+2q+1) \\
& \{86^24^22\} \\
57,70,[567] & +q^5(q^2+q+1)^3(q^2-q+1) \\
& (q^{12}+3q^{10}+q^9+3q^8+5q^6+3q^4+q^3+3q^2+1)\{865^242\} \\
58,69,[324] & +q^5(q^4+1)^2(q^2-q+1)^2(q^2+q+1)^4(\{865^23^2\}+\{7^25^242\}) \\
62,67,[405] & +q^5(q^4+q^3+q^2+q+1)(q^2+q+1)^4(q^2-q+1)^2(q^4-q^2+1) \\
& (\{864^4\}+\{6^442\}) \\
60,69,[450] & +q^5(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)(q^2+q+1)^2) \\
& \times(q^4+1)\{85^42\} \\
60,67,[270] & +q^5(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^4+1)(q^2-q+1) \\
& \times(q^2+q+1)^3(\{7^254^23\}+\{76^253^2\}) \\
61,66,[345] & +q^5(q^4+q^3+q^2+q+1)(q^2+q+1) \\
& \times(q^{14}+3q^{12}+4q^{10}-q^9+5q^8-q^7+5q^6-q^5+4q^4+3q^2+1) \\
& \{76^24^23\} \\
66,63,[225] & +q^5(q^4-q^3+q^2-q+1)(q^4-q^2+1)(q^2+q+1)^2 \\
& \times(q^4+q^3+q^2+q+1)^2\{6^34^3\} \\
61,70,[-1575] & -q^4(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)(q^2+q+1)^2 \\
& \times(q^6+q^5+q^4+q^3+q^2+q+1)(\{954^4\}+\{6^451\}) \\
61,67,[-900] & -q^4(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)(q^2+q+1)^2(q^2+1) \\
& \times(q^4+1)(\{85^343\}+\{765^32\}) \\
59,68,[-1215] & -q^4(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)(q^2+q+1)^5 \\
& \times(q^2-q+1)^2(\{8654^23\}+\{76^2542\}) \\
59,68,[-1422] & -q^4(q^4+1)(q^2+q+1)^2(q^{14}+2q^{13}+4q^{12}+4q^{11}+8q^{10}+7q^9 \\
& +10q^8+7q^7+10q^6+7q^5+8q^4+4q^3+4q^2+2q+1) \\
& \{7^25^23^2\} \\
64,64,[-720] & -q^4(q^4+q^3+q^2+q+1)(q^2+q+1)^2(q^4+1)(q^2+1)^3(q^4-q^2+1) \\
& (\{7654^3\}+\{6^3543\}) \\
63,66,[2250] & +q^3(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^3(q^4+1)(q^2+q+1)^2 \\
& (\{85^24^3\}+\{6^35^22\}) \\
62,65,[1800] & +q^3(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2(q^4+1)(q^2+q+1)^2 \\
& \times(q^2+1)^2\{765^243\} \\
65,64,[1050] & +q^3(q^6+q^5+q^4+q^3+q^2+q+1)(q^4+q^3+q^2+q+1)^2(q^2+q+1) \\
& \times(q^4+1)(q^4-q^3+q^2-q+1)\{75^43\}
\end{array}
$$

$$\begin{array}{ll} 67, 62, [945] & + q^3(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1) \\ & \times (q^4 - q^2 + 1)(q^2 + q + 1)^3(q^2 - q + 1)^2\{6^2 5^2 4^2\} \\ 66, 63, [-3150] & - q^2(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^4 - q^3 + q^2 - q + 1) \\ & \times (q^4 + 1)(q^2 + q + 1)^2(q^4 + q^3 + q^2 + q + 1)^2(\{75^3 4^2\} + \{6^2 5^3 3\}) \\ 70, 61, [4725] & + q(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^6 + q^3 + 1) \\ & \times (q^4 + q^3 + q^2 + q + 1)^2(q^2 + q + 1)^2(q^4 - q^3 + q^2 - q + 1)\{65^4 4\} \\ 75, 60, [-10395] & - (q^4 + q^3 + q^2 + q + 1)(q^6 + q^3 + 1) \\ & \times (q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1) \\ & (q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)^2\{5^6\} \end{array}$$

6. $q-$Sums for $N = 2...7$

We define the $q - Sum$ for the $q-$square of the Vandermonde as

$$QS(N) = \sum_{\lambda} c_\lambda(q) \tag{6-1}$$

and give below a list of their values as $q-$polynomials for N from 2 to 7. The rightmost entry is the value for $q = 1$

$$QS(2) = -(q^2 + 1) \qquad \text{(-2)}$$

$$QS(3) = -(q^6 + q^5 + 4q^4 + 2q^3 + 4q^2 + q + 1) \qquad \text{(-14)}$$

$$\begin{aligned} QS(4) = {} & +(q^{12} + 2q^{11} + 6q^{10} + 4q^9 + 11q^8 + 4q^7 + 14q^6 + 4q^5 \\ & + 11q^4 + 4q^3 + 6q^2 + 2q + 1) \qquad \text{(+70)} \end{aligned}$$

$$\begin{aligned} QS(5) = {} & +(q^{20} + 3q^{19} + 9q^{18} + 13q^{17} + 30q^{16} + 31q^{15} + 70q^{14} + 52q^{13} \\ & + 113q^{12} + 66q^{11} + 134q^{10} + 66q^9 + 113q^8 + 52q^7 + 70q^6 \\ & + 31q^5 + 30q^4 + 13q^3 + 9q^2 + 3q + 1) \qquad \text{(+910)} \end{aligned}$$

$$\begin{aligned} QS(6) = {} & -(q^{30} + 4q^{29} + 13q^{28} + 26q^{27} + 56q^{26} + 78q^{25} + 146q^{24} + 146q^{23} \\ & + 293q^{22} + 210q^{21} + 509q^{20} + 242q^{19} + 732q^{18} + 220q^{17} + 866q^{16} \\ & + 196q^{15} + 866q^{14} + 220q^{13} + 732q^{12} + 242q^{11} + 509q^{10} + 210q^9 \\ & + 293q^8 + 146q^7 + 146q^6 + 78q^5 + 56q^4 + 26q^3 + 13q^2 + 4q + 1) \\ & \qquad \text{(-7280)} \end{aligned}$$

$$\begin{aligned} QS(7) = {} & -(q^{42} + 5q^{41} + 18q^{40} + 44q^{39} + 100q^{38} + 178q^{37} + 336q^{36} \\ & + 490q^{35} + 874q^{34} + 1059q^{33} + 1929q^{32} + 1861q^{31} + 3721q^{30} \\ & + 2676q^{29} + 6386q^{28} + 3232q^{27} + 9780q^{26} + 3374q^{25} + 13130q^{24} \\ & + 3199q^{23} + 15237q^{22} + 3060q^{21} + 15237q^{20} + 3199q^{19} + 13130q^{18} \\ & + 3374q^{17} + 9780q^{16} + 3232q^{15} + 6386q^{14} + 2676q^{13} + 3721q^{12} \\ & + 1861q^{11} + 1929q^{10} + 1059q^9 + 874q^8 + 490q^7 + 336q^6 + 178q^5 \\ & + 100q^4 + 44q^3 + 18q^2 + 5q + 1) \qquad \text{(-138320)} \end{aligned}$$

The rightmost entry is the value for $q = 1$. Let us call $CS(N)$ the sum of the coefficients $c_\lambda(1)$. Then we observe that

$$\frac{CS(N)}{CS(N-1)} = (-1)^{N+1} \begin{cases} \dfrac{3N-2}{2} & N \text{ even} \\ 3N-2 & N \text{ odd} \end{cases} \tag{6-2}$$

and conjecture that

$$CS(N) = \prod_{x=0}^{[N/2]} (-3x+1) \prod_{x=0}^{[(N-1)/2]} (6x+1) \tag{6-3}$$

We recall that explicit calculation yields

N	Number of Partitions	$CS(N)$	Absolute Sum of Coefficients
2	2	-2	4
3	5	-14	28
4	16	70	292
5	59	910	4102
6	247	-7280	73444
7	1111	-138320	1605838
8	5294	1521520	41603200
9	26310	38038000	1247676262
10	135281	-532532000	42551137984

7. Basic $q-$polynomials

Suppose for N there exists a set of $\{\lambda\}$ associated with $c_\lambda(1) = (-1)^\phi x$ where x is the multiplicity and a $q-$polynomial of the form $(-1)^\phi q^p Q(q)$ then under $N \to N+1$ we conjecture that

$$\phi \to \phi, \quad p \to p+N, \quad Q(q) \to Q(q), \quad \{\lambda\} \to \{2N-2, \lambda\} \tag{7-1}$$

with any additional $\{\mu\}$ added so that the resultant list of $S-$functions is reversal invariant. In the Table 7-1 below the first column is the $S-$function multiplicity, the second column gives the smallest value of N associated with that multiplicity followed by the corresponding value of $(-1)^\phi q^p$ and the invariant $Q(q)$ polynomial and finally the set of $S-$functions associated with the minimal values.

Table 7-1 Some basic $q-$polynomials

$c_\lambda[1]$	N_{min}	$(-1)^\phi q^p$	$Q(q)$	λ
[1]	2	$+q$	1	$\{2\}$
[−3]	2	-1	(q^2+q+1)	$\{1^2\}$
[6]	3	$+q$	$(q^2+q+1)(q^2+1)$	$\{321\}$
[−15]	3	-1	$(q^2+q+1)(q^4+q^3+q^2+q+1)$	$\{2^3\}$
[9]	4	$+q^4$	$(q^2+q+1)^2$	$\{5^21^2\}$
[−12]	4	$-q^3$	$(q^2+q+1)(q^2+1)^2$	$\{5421\}$
[−9]	4	$-q^3$	$(q^2+q+1)^2(q^2-q+1)$	$\{53^21\}$
[−6]	4	$-q^3$	$(q^2+q+1)(q^4+1)$	$\{4^22^2\}$
[27]	4	$+q^2$	$(q^2+q+1)^3(q^2-q+1)$	$\{532^2\}+\{4^231\}$
[−45]	4	$-q$	$(q^2+q+1)^2(q^2-q+1)(q^4+q^3+q^2+q+1)$	$\{43^22\}$
[105]	4	$+1$	$(q^2+q+1)(q^4+q^3+q^2+q+1)$	
			$(q^6+q^5+q^4+q^3+q^2+q+1)$	$\{3^4\}$
[−18]	5	$-q^7$	$(q^2+q+1)^2(q^2+1)$	$\{7^2321\}+\{7651^2\}$
[24]	5	$+q^6$	$(q^2+q+1)(q^2+1)^3$	76421}
$[18]_1$	5	$+q^6$	$(q^2+q+1)^2(q^2-q+1)(q^2+1)$	$\{783^21\}+\{75^21\}$
$[45]_1$	5	$+q^6$	$(q^2+q+1)^2(q^4+q^3+q^2+q+1)$	$\{7^22^3\}+\{6^31^2\}$
[−54]	5	$-q^5$	$(q^2+q+1)^3(q^2-q+1)(q^2+1)$	$\{7632^2\}+\{6^2431\}$
$[-36]_1$	5	$-q^5$	$(q^2+q+1)^2(q^2-q+1)(q^2+1)^2$	$\{75431\}+\{74^31\}$

Table 7-1 (contd)

$c_\lambda[1]$	N_{min}	$(-1)^\phi q^p$	$Q(q)$	λ
$[-27]$	5	$-q^5$	$(q^2+q+1)^3(q^2-q+1)^2$	$\{7542^2\}+\{6^2431\}$
$[-18]_2$	5	$-q^5$	$(q^2+q+1)^2(q^2-q+1)(q^4+1)$	$\{6^23^22\}+\{65^231\}$
$[81]$	5	$+q^4$	$(q^2+q+1)^4(q^2-q+1)^2$	$\{753^22\}+\{65^231\}$
$[72]$	5	$+q^4$	$(q^2+q+1)^2(q^2+1)^2(q^4+1)$	$\{74^232\}+\{654^21\}$
$[111]$	5	$+q^4$	$(q^2+q+1)(q^{10}+2q^9+4q^8+3q^7+6q^6$ $+5q^5+6q^4+3q^3+4q^2+2q+1)$	$\{6^242^2\}$
$[45]_2$	5	$+q^4$	$(q^2+q+1)^2(q^4+q^3+q^2+q+1)$ $\times(q^4-q^3+q^2-q+1)$	$\{653^3\}+\{5^332\}$
$[-180]$	5	$-q^3$	$(q^2+q+1)^2(q^2+1)(q^4+q^3+q^2+q+1)$ $\times(q^4+1)$	$\{743^3\}+\{5^341\}$
$[-144]$	5	$-q^3$	$(q^2+q+1)^2(q^2+1)^3(q^4+1)$	$\{65432\}$
$[-90]$	5	$-q^3$	$(q^2+q+1)^2(q^2-q+1)(q^4+1)$ $\times(q^4+q^3+q^2+q+1)$	$\{64^32\}$
$[-75]$	5	$-q^3$	$(q^2+q+1)(q^4+q^3+q^2+q+1)^2$ $(q^4-q^3+q^2-q+1)$	$\{6^243^2\}$
$[270]$	5	$+q^2$	$(q^2+q+1)^3(q^2-q+1)(q^4+q^3+q^2+q+1)$ $\times(q^4+1)$	$\{64^23^2\}+\{5^24^22\}$
$[-420]$	5	$-q$	$(q^2+q+1)(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)$ $\times(q^6+q^5+q^4+q^3+q^2+q+1)$	$\{54^33\}$
$[945]$	5	$+1$	$(q^2+q+1)^2(q^4+q^3+q^2+q+1)$ $(q^6+q^5+q^4+q^3+q^2+q+1)(q^6+q^3+1)$	$\{4^5\}$

8. $N=8$ $q-$polynomials for admissible partitions with $c_\lambda(1)=0$

There are eight admissible partitions for $N = 8$ whose $S-$functions occur in expansion of the square of the Vandermonde with zero coefficient. They occur as pairs of partitions related by Dunne's reversal symmetry as

$$\{13\ 11985^241\},\quad \{13\ 10\ 9^26531\} \tag{8-1a}$$

$$\{13\ 11\ 9854^22\},\quad \{12\ 10^2\ 96531\} \tag{8-1b}$$

$$\{13\ 11\ 976541\},\quad \{13\ 10\ 987531\} \tag{8-1c}$$

$$\{12\ 11\ 97^24^22\},\quad \{12\ 10^2\ 7^2532\} \tag{8-1d}$$

Their corresponding $q-$polynomials are

$$-q^{17}(q^2-q+1)^2(q^2+1)^2(q-1)^4(q^2+q+1)^5 \tag{8-1a$'$}$$

$$+q^{16}(q^2+1)(q^2-q+1)^3(q-1)^4(q^2+q+1)^6 \tag{8-1b$'$}$$

$$+q^{16}(q^2-q+1)^2(q^2+1)^3(q-1)^4(q^2+q+1)^5 \tag{8-1c$'$}$$

$$+q^{14}(q^{10}+q^9+3q^8+4q^6+q^5+4q^4+3q^2+q+1)$$

$$\times(q^2-q+1)^2(q-1)^4(q^2+q+1)^5 \tag{8-1d$'$}$$

Some $q-$expansions

I had suggested earlier that if a $q-$polynomial is of the form $(-1)^\phi q^p Q(q)$ then under $N \rightarrow N+1$

$$\phi \rightarrow \phi, \quad p \rightarrow p+N, \quad Q(q) \rightarrow Q(q), \quad \{\lambda\} \rightarrow \{2N-2, \lambda\} \tag{9-1}$$

with an additional $\{\mu\}$ added to make the resultant list of $S-$functions reversal invariant. Below I give some counterexamples to the last part of the conjecture. I indicate self-reversal $S-$functions with a subscript 's'. It is important to check that each list of $S-$functions is associated with a common $q-$polynomial as seen, for example, for $[-27]_1$ and $[-27]_2$.

[9]

N

4 $\{5^2 1^2\}_s$

$+q^4(q^2+q+1)^2$

5 $(\{85^2 1^2\}+\{7^2 3^2\})+\{7^2 41^2\}_s$

$+q^8(q^2+q+1)^2$

6 $(\{10\ 85^2 1^2\}+\{9^2 5^2 2\})+\{10\ 7^2 3^2\}_s+(\{10\ 7^2 41^2\}+\{9^2 63^2\})+\{9^2 641^2\}_s$

$+q^{13}(q^2+q+1)^2$

7 $(12\ 10\ 85^2 1^2\}+\{11^2\ 7^2 42\})+(\{12\ 10\ 7^2 3^2\}+\{12\ 9^2 5^2 2\})$

$+(\{12\ 10\ 7^2 41^2\}+\{11^2\ 85^2 2\})+(\{12\ 9^2 63^2\}+\{11^2\ 863^2 1\})$

$+\{12\ 9^2 63^2\}_s+\{11^2 8641^2\}_s$

$+q^{19}(q^2+q+1)^2$

[−18]

N

5 $(\{7^2 321\}+\{7651^2\})$

$-q^7(q^2+1)(q^2+q+1)^2$

6 $(\{10\ 7^2 321\}+\{9873^2\})+(\{10\ 7651^2\}+\{9^2 543\})+\{98741^2\}_s$

$-q^{12}(q^2+1)(q^2+q+1)^2$

7 $(\{12\ 10\ 7^2 321\}+\{11\ 10\ 95^2 2\})+(\{12\ 9873^2\}+\{12\ 9^2 543\})$

$+(\{12\ 10\ 7651^2\}+\{11^2\ 7652\})+(\{12\ 9^2 6321\}+\{11\ 10\ 963^2\})$

$+(\{12\ 98741^2\}+\{11^2\ 8543\})+(\{11^2\ 86321\}+\{11\ 10\ 9641^2\})$

$-q^{18}(q^2+1)(q^2+q+1)^2$

$[-27]_1$

N

5 $(\{7542^2\}+\{6^2431\})$
$-q^5(q^2-q+1)^2(q^2+q+1)^3$

6 $(\{10\ 7542^2\}+\{8^2653\})+(\{10\ 6^2431\}+\{9764^2\})+\{97^23^21\}_s$
$-q^{10}(q^2-q+1)^2(q^2+q+1)^3$

7 $(\{12\ 10\ 7542^2\}+\{10^2\ 8752\})+(\{12\ 10\ 6^2431\}+\{11\ 986^22\})$
$+(\{12\ 9764^2\}+\{12\ 8^2653\})+(\{12\ 97^23^21\}+\{11\ 9^25^23\})$
$-q^{16}(q^2-q+1)^2(q^2+q+1)^3$

$[-27]_2$

N

6 $\{9^25^21^2\}_s$
$-q^{12}(q^2+q+1)^3$

7 $(\{12\ 9^25^21^2\}+\{11^2\ 7^23^2\})+(\{11^2\ 85^21^2\}+\{11^2\ 7^241^2\})$
$-q^{18}(q^2+q+1)^3$

9. The complete $q-$polynomials for $N=7$

[1] $\{12\ 10\ 8642\}$
$+q^{21}$

[6] $\{12\ 10\ 86321\},\{12\ 10\ 8543\},\{12\ 10\ 7652\},\{12\ 98742\},\{11\ 10\ 9642\}$
$+q^{19}(q^2+q+1)(q^2+1)$

[9] $\{12\ 10\ 85^21^2\},\{12\ 10\ 7^241^2\},\{12\ 10\ 7^23^2\},\{12\ 9^2641^2\},\{12\ 9^263^2\}$
$\{12\ 9^25^22\},\{11^2\ 7^242\},\{11^2\ 85^22\},\{11^2\ 863^2\},\{11^2\ 8641^2\}$
$+q^{19}(q^2+q+1)^2$

$[18]_1$ $\{12\ 10\ 763^21\},\{12\ 10\ 75^221\},\{12\ 985^23\},\{12\ 97^243\},\{11\ 10\ 7^252\}$
$+q^{17}(q^2+q+1)^2(q^2+1)(q^2-q+1)$

$[18]_2$ $\{12\ 8^26^21^2\},\{12\ 9^24^22^2\},\{11^2\ 84^22^2\},\{11^2\ 6^24^2\}$
$\{11\ 9^2652\},\{10^2\ 8^241^2\},\{10^2\ 8^23^2\}$
$+q^{17}(q^2+q+1)^2(q^4+1)$

[24] $\{12\ 10\ 76421\},\{12\ 98643\},\{11\ 10\ 8652\}$
$+q^{17}(q^2+q+1)(q^2+1)^3$

[27] $\{12\ 10\ 8532^2\}, \{12\ 10\ 84^231\}, \{12\ 8^2752\}, \{12\ 10\ 754^2\}$
$\{12\ 10\ 6^253\}, \{12\ 9^253^21\}, \{12\ 97^251^2\}, \{12\ 976^22\}$
$\{11^2\ 853^21\}, \{11^2\ 75^23\}, \{11\ 9^2741^2\}, \{11\ 9^273^2\}$
$\{11\ 98^242\}, \{10^2\ 9742\}$
$+ q^{17}(q^2+q+1)^3(q^2-q+1)$

[36] $\{12\ 9^25421\}, \{12\ 987321\}, \{12\ 98651^2\}, \{11^2\ 85421\}$
$\{11^2\ 7643\}, \{11\ 10\ 96321\}, \{11\ 10\ 9543\}, \{11\ 10\ 8741^2\}$
$\{11\ 10\ 873^2\}$
$+ q^{17}(q^2+q+1)^2(q^2+1)^2$

$[45]_1$ $\{12\ 10\ 7^22^3\}, \{12\ 10\ 6^31^2\}, \{12\ 9^262^3\}, \{12\ 9^24^3\}$
$\{12\ 8^341^2\}, \{12\ 8^33^2\}, \{10^3\ 641^2\}, \{10^3\ 63^2\}, \{10^3\ 5^22\}$
$+ q^{17}(q^2+q+1)^2(q^4+q^3+q^2+q+1)$

$[45]_2$ $\{12\ 10\ 653^3\}, \{12\ 10\ 5^332\}, \{12\ 875^3\}, \{12\ 7^354\}, \{10\ 97^32\}, \{9^3762\}$
$+ q^{15}(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$

$[45]_3$ $\{12\ 7^24^4\}, \{12\ 6^43^2\}, \{11^2\ 862^3\}, \{11^2\ 84^3\}, \{11^2\ 6^32\}, \{9^26^4\}, \{8^45^2\}$
$+ q^{13}(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$
$\times (q^4-q^2+1)$

$[54]_1$ $\{12\ 98542^2\}, \{12\ 8^26521\}, \{11^2\ 6^23^22\}, \{11^265^22^2\}, \{11\ 10\ 7^23^21\}$
$\{11\ 10\ 764^2\}, \{11\ 9^263^21\}, \{11\ 9^25^221\}$
$\{10^2\ 8743\}, \{10^2\ 7^261^2\}, \{10\ 9^26^21^2\}$
$+ q^{15}((q^2+q+1)^3(q^2+1)(q^2-q+1)^2$

$[54]_2$ $\{11^2\ 7^2321\}, \{11^2\ 7651^2\}, \{11\ 10\ 95^21^2\}$
$+ q^{17}(q^2+q+1)^3(q^2+1)$

$[54]_3$ $\{11\ 96^2532\}, \{10\ 976^231\}$
$+ q^{11}(q^2+q+1)^3(q^2-q+1)$
$\times (q^{12}-2q^{11}+3q^{10}-5q^9+7q^8-7q^7+8q^6-7q^5+7q^4-5q^3$
$+ 3q^2-2q+1)$

$[72]_1$ $\{12\ 10\ 74^232\}, \{10\ 98^252\}, \{12\ 10\ 654^21\}, \{11\ 8^2762\}, \{12\ 984^31\}$
$\{11\ 8^343\}, \{12\ 96^321\}, \{11\ 10\ 6^33\}, \{12\ 96^254\}, \{12\ 876^23\}$
$+ q^{15}(q^2+q+1)^2(q^2+1)^2(q^4+1)$

$[72]_2$ $\{12\ 985431\}, \{11\ 98743\}, \{12\ 976521\}, \{11\ 10\ 7653\}, \{11\ 10\ 863^21\}$
$\{11\ 9^26421\}, \{11\ 10\ 85^221\}, \{11\ 10\ 7^2421\}$
$+ q^{15}(q^2+q+1)^2(q^2-q+1)(q^2+1)^3$

$[81]_1$ $\{12\ 10\ 753^22\}, \{10\ 9^2752\}, \{12\ 10\ 65^231\}, \{11\ 97^262\}, \{12\ 97^232^2\}$
$\{10^2\ 95^23\}, \{12\ 975^24\}, \{12\ 87^253\}, \{12\ 975^22^2\}, \{10^2\ 7^253\}$
$\{12\ 8^273^21\}, \{11\ 9^254^2\}, \{11^2\ 7542^2\}, \{10^2\ 8751^2\}, \{11^2\ 6^2431\}$
$\{11\ 986^21^2\}$
$+ q^{15}(q^2+q+1)^4(q^2-q+1)^2$

$[81]_2$ $\{12\ 8^25^231\}, \{11\ 97^24^2\}$
$+ q^{15}(q^2+q+1)^4(q^2-q+1)(2q^2-3q+2)$

$[81]_3$ $\{10\ 8^273^3\}, \{9^354^22\}, \{11\ 7^34^22\}, \{10\ 8^25^31\}$
$+ q^{11}(q^2+q+1)^4(q^{10}-2q^9+2q^8-2q^7+3q^6-3q^5+3q^4$
$-2q^3+2q^2-2q+1)$

$[81]_4$ $\{11\ 985^231\}, \{11\ 97^2431\}$
$+ q^{13}(q^2+q+1)^3(q^2-q+1)(q^8-2q^7+4q^6-5q^5+7q^4$
$-5q^3+4q^2-2q+1)$

$[90]_1$ $\{12\ 95^33^2\}, \{9^27^33\}, \{12\ 7^25^31\}, \{11\ 7^35^2\}$
$+ q^{13}(q^2+q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$

$[90]_2$ $\{10^3\ 4^22^2\}, \{10^2\ 8^22^3\}$
$+ q^{15}(q^2+q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)$

$[96]$ $\{11\ 10\ 86421\}$
$+ q^{15}(q^2+q+1)(q^2+1)^5$

$[99]$ $\{12\ 976431\}, \{11\ 98653\}$
$+ q^{15}(q^2+q+1)^2(q^2-q+1)(2q^6+4q^4-q^3+4q^2+2)$

$[105]$ $\{12\ 10\ 83^4\}, \{9^442\}, \{12\ 10\ 5^4\}, \{12\ 7^42\}$
$+ q^{15}(q^2+q+1)(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$

$[108]_1$ $\{12\ 98632^2\}, \{12\ 8^27421\}, \{11^2\ 75431\}, \{11\ 10\ 854^2\}, \{11\ 98751^2\}$
$+ q^{15}(q^2+q+1)^3(q^2-q+1)(q^2+1)^2$

$[108]_2$ $\{11^2\ 74^31\}, \{11\ 8^351^2\}, \{10^2\ 9643\}$
$+ q^{15}(q^2+q+1)^3(q^2+1)(q^4+1)$

$[108]_3$ $\{12\ 965^232\}, \{12\ 7^26^22^2\}, \{10^2\ 6^25^2\}, \{9^28^24^2\}$
$+ q^{13}(q^2+q+1)^3(q^2-q+1)^2(q^2+1)(q^4+1)$

$[108]_4$ $\{12\ 8^24^23^2\}, \{12\ 875^241\}, \{11\ 87^254\}, \{10\ 97^263\}$
$+ q^{13}(q^2+q+1)^3(q^2-q+1)(q^4+1)^2$

$[111]$ $\{12\ 10\ 6^242^2\}, \{12\ 8^264^2\}, \{10^2\ 86^22\}$
$+ q^{15}(q^2+q+1)(q^(10)+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4$
$+3q^3+4q^2+2q+1)$

[135] $\{12\ 9^2 43^2 2\}, \{12\ 87^2 61^2\}, \{11^2\ 843^2 2\}, \{11^2\ 65^2 4\}$
$\{11\ 9^2 72^3\}, \{10^3\ 53^2 1\}, \{10\ 9^2 841^2\}, \{10\ 9^2 83^2\}$
$+ q^{15}(q^2+q+1)^3(q^2-q+1)(q^4+q^3+q^2+q+1)$

[153] $\{11\ 986431\}$
$+ q^{13}(q^2+q+1)^2(q^2-q+1)(q^{10}-q^9+5q^8-3q^7+9q^6-5q^5$
$+ 9q^4-3q^3+5q^2-q+1)$

$[162]_1$ $\{11^2\ 7632^2\}, \{11^2\ 6^2 521\}, \{11\ 10\ 9532^2\}, \{11\ 10\ 94^2 31\}$
$\{11\ 10\ 76^2 1^2\}, \{10^2\ 97321\}, \{10^2\ 9651^2\}$
$+ q^{15}(q^2+q+1)^4(q^2-q+1)(q^2+1)$

$[162]_2$ $\{12\ 975432\}, \{12\ 876531\}, \{11\ 10\ 6^2 531\}, \{11\ 10\ 754^2 1\}, \{11\ 10\ 763^2 2\}$
$\{11\ 98^2 321\}, \{10^2\ 7^2 431\}\{10\ 9^2 6521\}, \{10\ 98753\}, \{9^3 6^2 21\}$
$+ q^{13}(q^2+q+1)^4(q^2-q+1)^3(q^2+1)$

$[162]_3$ $\{12\ 9753^3\}, \{12\ 974^3 2\}$
$+ q^{13}(q^2+q+1)^4(q^2-q+1)^2(q^4+1)$

$[162]_4$ $\{12\ 8763^3\}, \{12\ 86^3 31\}, \{12\ 7^3 531\}, \{12\ 7^3 432\}, \{11\ 10\ 6^2 3^3\}$
$\{11\ 985^2 2^2\}, \{11\ 976^2 21\}, \{11\ 97654\}\{11\ 975^3\}, \{11\ 96^3 4\}$
$\{11\ 8^2 7521\}, \{10\ 985^3\}, \{10\ 8^3 53\}, \{9^3 753\}, \{9^3 654\}$
$+ q^{13}(q^2+q+1)^3(q^2+1)(q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$

$[162]_5$ $\{10\ 7^3 53^2\}, \{9^2 75^3 2\}$
$+ q^9(q^2+q+1)^4(q^4+1)(q^{10}-2q^9+2q^8-2q^7+3q^6-3q^5+3q^4$
$- 2q^3+2q^2-2q+1)$

$[180]_1$ $\{11\ 10\ 872^3\}, \{10^3\ 5421\}$
$+ q^{15}(q^2+q+1)^2(q^2+1)^2(q^4+q^3+q^2+q+1)$

$[180]_2$ $\{10^2\ 5^4 2\}, \{10\ 7^4 2^2\}$
$+ q^{11}(q^2+q+1)^2(q^4+1)^2(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$

[216] $\{12\ 965^2 41\}, \{11\ 9^2 4^2 32\}, \{11\ 87^2 63\}, \{10\ 98^2 3^2 1\}$
$+ q^(13)(q^2+q+1)^3(q^2-q+1)(q^2+1)^2(q^4+1)$

$[225]_1$ $\{12\ 8^3 2^3\}, \{10^3\ 62^3\}, \{10^3\ 4^3\}, \{9^2 87^2 1^2\}$
$+ q^{15}(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2$

$[225]_2$ $\{12\ 95^4 1\}, \{11^2\ 5^2 43^2\}, \{11\ 7^4 3\}$
$+ q^{13}(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)$

$[225]_3$ $\{12\ 6^3 4^3\}, \{8^3 6^3\}$
$+ q^{11}(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)$
$\times (q^4-q^2+1)$

$[225]_4$ $\{9^25^44\}, \{87^43^2\}$
$+ q^9(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)^2$
$\times (q^4-q^2+1)$

$[234]$ $\{12\ 96^2432\}, \{12\ 8^25432\}, \{12\ 87652^2\}, \{12\ 8764^21\}, \{11\ 10\ 6^252^2\}$
$\{11\ 10\ 6^24^21\}, \{11\ 8^26^221\}\{11\ 8^2654\}, \{10^2\ 76^221\}, \{10^2\ 7654\}$
$\{10\ 9874^2\}, \{10\ 986^23\}$
$+ q^{13}(q^2+q+1)^2(q^2+1)(q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)$

$[243]_1$ $\{11\ 9^253^22\}, \{10^2\ 9542^2\}, \{10^2\ 8732^2\}, \{10\ 9^273^21\}$
$+ q^{13}(q^2+q+1)^5(q^2-q+1)^3$

$[243]_2$ $\{11\ 97^23^22\}, \{10\ 9^25^231\}$
$+ q^{13}(q^2+q+1)^5(q^2-q+1)^2(2q^2-q+1)^2$

$[243]_3$ $\{11\ 96^32^2\}, \{10^2\ 6^331\}$
$+ q^{13}(q^2+q+1)^4(q^2-q+1)(2q^6-2q^5+2q^4-q^3+2q^2-2q+2)$

$[243]_4$ $\{11\ 8^254^22\}, \{10\ 8^274^21\}$
$+ q^{11}(q^2+q+1)^4(q^2-q+1)$
$(q^{10}-2q^9+4q^8-5q^7+6q^6-5q^5+6q^4-5q^3+4q^2-2q+1)$

$[243]_5$ $\{10\ 8^254^23\}, \{98^274^22\}$
$+ q^9(q^2+q+1)^5(q^2-q+1)^2(q^{10}-2q^9+2q^8-2q^7+3q^6-3q^5$
$+ 3q^4-2q^3+2q^2-2q+1)$

$[270]_1$ $\{12\ 10\ 64^23^2\}, \{12\ 10\ 5^24^22\}, \{12\ 86^25^2\}, \{12\ 7^26^24\}$
$\{11^2\ 64^32\}, \{10\ 8^361^2\}, \{10\ 8^27^22\}, \{9^28^262\}$
$+ q^{13}(q^2+q+1)^3(q^2-q+1)(q^4+q^3+q^2+q+1)(q^4+1)$

$[270]_2$ $\{12\ 7^254^23\}, \{12\ 76^253^2\}, \{9^276^25\}, \{98^275^2\}$
$+ q^{11}(q^2+q+1)^3(q^2-q+1)(q^4+q^3+q^2+q+1)$
$(q^4-q^3+q^2-q+1)(q^4+1)$

$[279]$ $\{10^2\ 85^22^2\}, \{10^2\ 7^242^2\}$
$+ q^{13}(q^2+q+1)^2(q^2-q+1)(2q^{10}+2q^9+4q^8+6q^6+3q^5+5q^4$
$+ 4q^2+2q+2)$

$[288]$ $\{11\ 10\ 84^232\}, \{10\ 98^2421\}$
$+ q^{13}(q^2+q+1)^2(q^2+1)^4(q^4+1)$

$[297]$ $\{10^2\ 75^232\}, \{10\ 97^252^2\}$
$+ q^{11}(q^2+q+1)^3(q^2-q+1)$
$(q^{12}-q^{11}+3q^{10}-3q^9+7q^8-5q^7+7q^6-5q^5+7q^4-3q^3+3q^2-q+1)$

$[324]_1$ $\{11\ 10\ 7642^2\}, \{11\ 10\ 75^231\}, \{11\ 97^2521\}, \{10^2\ 86521\}$
$+ q^{13}(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^6+3q^4-2q^3+3q^2+1)$

$[324]_2$ $\{11\ 10\ 853^22\}, \{11\ 98732^2\}, \{10^2\ 95431\}, \{10\ 9^27421\}$
$+ q^{13}(q^2+q+1)^4(q^2-q+1)^2(q^2+1)^2$

$[324]_3$ $\{11\ 9854^21\}, \{11\ 8^27431\}$
$+ q^{13}(q^2+q+1)^4(q^2-q+1)(q^2+1)^2(2q^2-3q+2)$

$[324]_4$ $\{11\ 8^332^2\}, \{10^2\ 94^31\}$
$+ q^{13}(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4+1)$

$[324]_5$ $\{12\ 865^23^2\}, \{12\ 7^25^242\}, \{10\ 9^24^23^2\}, \{10\ 87^25^2\}, \{9^28^23^22\}, \{9^27^264\}$
$+ q^{11}(q^2+q+1)^4(q^2-q+1)^2(q^4+1)^2$

$[324]_6$ $\{11\ 8^24^33\}, \{98^34^21\}$
$+ q^{11}(q^2+q+1)^3(q^2+1)(q^4+1)(q^8-q^7+2q^6-q^5+q^4$
$-q^3+2q^2-q+1)$

$[324]_7$ $\{11\ 87^2432\}, \{10\ 985^241\}$
$+ q^{11}(q^2+q+1)^3(q^2+1)^2(q^{10}-2q^9+4q^8-5q^7+6q^6-5q^5+6q^4$
$-5q^3+4q^2-2q+1)$

$[333]$ $\{12\ 8^263^22\}, \{12\ 87^242^2\}, \{10^2\ 85^24\}, \{10\ 9^264^2\}$
$+ q^{13}(q^2+q+1)^2(q^2-q+1)(q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4$
$+3q^3+4q^2+2q+1)$

$[345]$ $\{12\ 76^24^23\}, \{98^26^25\}$
$+ q^{11}(q^2+q+1)(q^4+q^3+q^2+q+1)(q^{14}+3q^{12}+4q^{10}-q^9+5q^8$
$-q^7+5q^6-q^5+4q^4+3q^2+1)$

$[351]$ $\{11\ 98642^2\}, \{10^2\ 86431\}$
$+ q^{13}(q^2+q+1)^3(q^2-q+1)^2(2q^6+q^5+4q^4-q^3+4q^2+q+2)$

$[360]_1$ $\{12\ 9843^3\}, \{12\ 7^3621\}, \{11\ 10\ 65^3\}, \{9^3843\}$
$+ q^{13}(q^2+q+1)^2(q^2+1)^2(q^4+q^3+q^2+q+1)(q^4+1)$

$[360]_2$ $\{9^34^33\}, \{98^33^3\}$
$+ q^{11}(q^2+q+1)^2(q^2+1)(q^4+q^3+q^2+q+1)(q^4+1)$
$(q^6+q^4-2q^3+q^2+1)$

$[378]_1$ $\{10\ 97^23^3\}, \{9^35^232\}$
$+ q^{11}(q^2+q+1)^3(q^4+1)(q^{10}-q^9+4q^8-4q^7+6q^6-5q^5+6q^4$
$-4q^3+4q^2-q+1)$

$[378]_2$ $\{10\ 975^2 3^2\}, \{9^2 7^2 532\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)(q^4+1)(q^{12}-q^{11}+2q^{10}-3q^9+6q^8$
$-5q^7+7q^6-5q^5+6q^4-3q^3+2q^2-q+1)$

$[387]$ $\{11\ 7^3 43^2\}, \{9^2 85^3 1\}$
$+ q^{11}(q^2+q+1)^2(q^{16}+4q^{14}+7q^{12}-q^{11}+9q^{10}-3q^9+9q^8$
$-3q^7+9q^6-q^5+7q^4+4q^2+1)$

$[396]_1$ $\{11\ 8^2 6432\}, \{10\ 9864^2 1\}, \{10^2\ 76432\}, \{1098652^2\}$
$+ q^{11}(q^2+q+1)^2(q^2+1)^2(q^{12}-q^{11}+3q^{10}-3q^9+7q^8$
$-5q^7+7q^6-5q^5+7q^4-3q^3+3q^2-1+1)$

$[396]_2$ $\{10^2\ 853^3\}, \{9^3 742^2\}$
$+ q^{11}(q^2+q+1)^2(q^4+1)(q^{12}+2q^{10}+5q^8+6q^6+5q^4+2q^2+1)$

$[405]_1$ $\{12864^4\}, \{126^4 42\}, \{1086^4\}, \{8^4 64\}$
$+ q^{11}(q^2+q+1)^4(q^2-q+1)^2(q^4-q^2+1)(q^4+q^3+q^2+q+1)$

$[405]_2$ $\{108^2 4^4\}, \{8^4 4^2 2\}$
$+ q^{11}(q^2+q+1)^4(q^2-q+1)(q^4+q^3+q^2+q+1)$
$\times (2q^6-3q^5+q^4+q^3+q^2-3q+2)$

$[432]_1$ $\{11^2\ 65432\}, \{1098761^2\}$
$q^{13}(q^2+q+1)^3(q^2+1)^3(q^4+1)$

$[432]_2$ $\{11\ 965^3 1\}, \{11\ 7^3 631\}$
$q^{11}(q^2+q+1)^3(q^2+1)^2(q^2-q+1)(q^4+1)^2$

$[450]_1$ $\{1285^4 2\}, \{10\ 7^4 4\}$
$q^{11}(q^2+q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)$

$[450]_2$ $\{1176^2 4^3\}, \{8^3 6^2 51\}, \{10\ 95^4 3\}, \{97^4 32\}$
$q^9(q^2+q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)^2$

$[486]_1$ $\{11984^2 3^2\}, \{9^2 8^2 431\}$
$q^{11}(q^2+q+1)^5(q^2-q+1)^3(q^4+1)$

$[486]_2$ $\{11\ 975^2 32\}, \{10\ 97^2 531\}$
$q^{11}(q^2+q+1)^5(q^2-q+1)^2(q^6-2q^5+5q^4-6q^3+5q^2-2q+1)$

$[486]_3$ $\{11\ 875^3 1\}, \{11\ 7^3 541\}$
$q^{11}(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^8-q^7+2q^6-q^5$
$+q^4-q^3+2q^2-q+1)$

$[486]_4$ $\{10^2\ 9632^2\}$
$q^{13}(q^2+q+1)^5(q^2-q+1)^2(q^2+1)$

$[486]_5$ $\{10^2\ 65^23^2\}, \{9^27^262^2\}$
$q^{11}(q^2+q+1)^4(q^2-q+1)(q^4+1)(2q^6-2q^5+2q^4-q^3+2q^2-2q+2)$

$[540]$ $\{11^2\ 743^3\}, \{11^2\ 5^341\}, \{11\ 87^31^2\}, \{9^3851^2\}$
$q^{13}(q^2+q+1)^3(q^2+1)(q^4+1)(q^4+q^3+q^2+q+1)$

$[567]_1$ $\{12\ 865^242\}, \{10\ 87^264\}$
$q^{11}(q^2+q+1)^3(q^2-q+1)(q^{12}+3q^{10}+q^9+3q^8+5q^6+3q^4$
$+q^3+3q^2+1)$

$[567]_2$ $\{11\ 8^25^232\}, \{10\ 97^24^21\}$
$q^{11}(q^2+q+1)^3(q^{14}-q^{13}+5q^{12}-6q^{11}+13q^{10}-13q^9$
$+19q^8-15q^7+19q^6-13q^5+13q^4-6q^3+5q^2-q+1)$

$[576]$ $\{11\ 10\ 6543^2\}, \{9^287621\}$
$q^{11}(q^2+q+1)^2(q^2+1)^4(q^4+1)^2$

$[630]$ $\{11\ 10\ 93^4\}, \{9^4321\}$
$q^{13}(q^2+1)(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$

$[648]_1$ $\{11\ 10\ 654^22\}, \{10\ 8^27621\}$
$q^{11}(q^2+q+1)^4(q^2-q+1)^2(q^2+1)^2(q^4+1)$

$[648]_2$ $\{11\ 976432\}, \{10\ 986531\}$
$q^{11}(q^2+q+1)^4(q^2-q+1)(q^2+1)^2(q^6-2q^5+5q^4$
$-6q^3+5q^2-2q+1)$

$[648]_3$ $\{11\ 8^263^3\}, \{10^2\ 763^3\}, \{9^364^21\}, \{9^3652^2\}$
$q^{11}(q^2+q+1)^3(q^2+1)(q^4+1)(q^8+2q^6-2q^5+4q^4-2q^3+2q^2+1)$

$[648]_4$ $\{11\ 86^332\}, \{10\ 96^341\}$
$q^{11}(q^2+q+1)^3(q^2+1)^2(q^4+1)(2q^6-2q^5+2q^4-q^3+2q^2-2q+2)$

$[648]_5$ $\{10^2\ 6^2532\}, \{10\ 976^22^2\}$
$q^{11}(q^2+q+1)^4(q^2-q+1)(q^{10}-q^9+5q^8-5q^7+6q^6-4q^5$
$+6q^4-5q^3+5q^2-q+1)$

$[666]$ $\{12\ 86^24^22\}, \{10^2\ 84^32\}, \{10\ 8^342^2\}, \{10\ 8^26^24\}$
$+q^{11}(q^2+q+1)^2(q^2-q+1)(q^4+1)(q^{10}+2q^9+4q^8+3q^7+6q^6$
$+5q^5+6q^4+3q^3+4q^2+2q+1)$

$[675]_1$ $\{12\ 95^24^23\}, \{12\ 76^25^21\}, \{11\ 7^26^25\}, \{98^27^23\}$
$+q^{11}(q^2+q+1)^3(q^2-q+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$

$[675]_2$ $\{10^3\ 43^22\}, \{10\ 9^282^3\}$
$+q^{13}(q^2+q+1)^3(q^2-q+1)(q^4+q^3+q^2+q+1)^2$

$[675]_3$ $\{8^35^24^2\}, \{8^27^24^3\}$
$+ q^9(q^2+q+1)^3(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$
$\times (2q^6-3q^5+q^4+q^3+q^2-3q+2)$

$[675]_4$ $\{8^275^34\}, \{87^354^2\}$
$+ q^7(q^2+q+1)^3(q^2-q+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1)^2$
$\times (q^4+q^3+q^2+q+1)^2$

$[702]$ $\{11\ 96^2541\}, \{11\ 876^231\}, \{10\ 9^25432\}, \{10\ 9873^22\}$
$+ q^{11}(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^{10}+3q^8-q^7+4q^6$
$-q^5+4q^4-q^3+3q^2+1)$

$[720]$ $\{12\ 964^33\}, \{12\ 76^341\}, \{11\ 86^35\}, \{98^363\}$
$+ q^{11}(q^2+q+1)^2(q^2+1)^2(q^4+1)^2(q^4+q^3+q^2+q+1)$

$[729]_1$ $\{11\ 9754^22\}, \{10\ 8^27531\}$
$+ q^{11}(q^2+q+1)^6(q^2-q+1)^3(2q^2-3q+2)$

$[729]_2$ $\{11\ 96^243^2\}, \{9^286^231\}$
$+ q^{11}(q^2+q+1)^5(q^2-q+1)^2(2q^6-2q^5+2q^4-q^3+2q^2-2q+2)$

$[729]_3$ $\{11\ 7^362^2\}, \{10^2\ 65^31\}$
$+ q^{11}(q^2+q+1)^5(q^2-q+1)(q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$

$[738]$ $\{9^285^23^2\}, \{9^27^243^2\}$
$+ q^9(q^2+q+1)^2(q^4+1)(q^{16}+3q^{14}-q^{13}+8q^{12}-3q^{11}$
$+11q^{10}-5q^9+13q^8-5q^7+11q^6-3q^5+8q^4-q^3+3q^2+1)$

$[756]$ $\{11\ 876541\}$
$+ q^{11}(q^2+q+1)^3(q^2+1)^2(2q^{10}-3q^9+5q^8-5q^7+8q^6-7q^5$
$+8q^4-5q^3+5q^2-3q+2)$

$[765]$ $\{98^254^3\}, \{8^374^23\}$
$+ q^9(q^2+q+1)^2(q^4+q^3+q^2+q+1)$
$(q^{16}-q^{15}+q^{14}-3q^{13}+7q^{12}-6q^{11}+9q^{10}-8q^9+11q^8$
$-8q^7+9q^6-6q^5+7q^4-3q^3+4q^2-1+1)$

$[774]_1$ $\{10^2\ 85432\}, \{10\ 98742^2\}$
$+ q^{11}(q^2+q+1)^2(q^2+1)(q^{14}+4q^{12}-q^{11}+9q^{10}-3q^9+13q^8$
$-3q^7+13q^6-3q^5+9q^4-q^3+4q^2+1)$

$[774]_2$ $\{10\ 7^34^23\}, \{98^25^32\}$
$+ q^9(q^2+q+1)^2(q^4+1)(q^{16}+4q^{14}+7q^{12}-q^{11}+9q^{10}$
$-3q^9+9q^8-3q^7+9q^6-q^5+7q^4+4q^2+1)$

$[801]_1$ $\{12\ 87543^2\}, \{12\ 7^2 6532\}, \{10\ 9765^2\}, \{9^2 8754\}$
$+ q^{11}(q^2+q+1)^2(q^{16}+q^{15}+4q^{14}+2q^{13}+9q^{12}+3q^{11}$
$+ 13q^{10}+4q^9+15q^8+4q^7+13q^6+3q^5+9q^4+2q^3+4q^2+q+1)$

$[801]_2$ $\{10^2\ 6^2 4^2 2\}, \{10 8^2 6^2 2^2\}$
$+ q^{11}(q^2+q+1)^2$
$(2q^{14}+3q^{13}+6q^{12}+3q^{11}+10q^{10}+7q^9+12q^8+3q^7$
$+ 12q^6+7q^5+10q^4+3q^3+6q^2+3q+2)$

$[810]_1$ $\{11\ 10\ 5^3 42\}, \{10\ 87^3 21\}$
$+ q^{11}(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[810]_2$ $\{11\ 76^3 42\}, \{10\ 86^3 51\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)^2(q^4+1)(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[810]_3$ $\{11\ 6^4 43\}, \{986^4 1\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[837]$ $\{10\ 96^2 542\}, \{10\ 876^2 32\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)(q^{16}-q^{15}+4q^{14}-3q^{13}+9q^{12}$
$- 6q^{11}+13q^{10}-9q^9+15q^8-9q^7+13q^6-6q^5+9q^4-3q^3+4q^2-q+1)$

$[864]$ $\{11\ 865^3 2\}, \{10\ 7^3 641\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)(q^2+1)^2(q^4+1)^3$

$[900]$ $\{10^2\ 5^2 4^3\}, \{8^3 7^2 2^2\}$
$+ q^9(q^2+q+1)^2(q^4+1)^2(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$

$[918]$ $\{10^2\ 5^2 4^3\}, \{8^3 7^2 2^2\}$
$+ q^{11}(q^2+q+1)^3(q^2-q+1)^2(q^2+1)(q^8+2q^7+4q^6+q^5+q^4$
$+ q^3+4q^2+2q+1)$

$[936]$ $\{11\ 86^2 4^2 3\}, \{98^2 6^2 41\}$
$+ q^9(q^2+q+1)^2(q^2+1)(q^4+1)^2(q^{10}+3q^8-q^7+4q^6-q^5$
$+ 4q^4-q^3+3q^2+1)$

$[945]_1$ $\{12\ 10\ 4^5\}, \{126^5\}, \{8^5 2\}$
$+ q^{11}(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^6+q^3+1)$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)$

$[945]_2$ $\{12\ 6^25^24^2\}, \{8^27^26^2\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)^2(q^4-q^2+1)(q^4+q^3+q^2+q+1)$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)$

$[945]_3$ $\{8^265^4\}, \{7^464^2\}$
$+ q^7(q^2+q+1)^3(q^2-q+1)(q^4-q^2+1)(q^4+q^3+q^2+q+1)$
$\times (q^6-q^5+q^4-q^3+q^2-q+1)$
$(q^6+q^5+q^4+q^3+q^2+q+1)$

$[972]_1$ $\{11\ 10\ 74^23^2\}, \{9^28^2521\}$
$+ q^{11}(q^2+q+1)^5(q^2-q+1)^2(q^2+1)(q^4+1)$

$[972]_2$ $\{10\ 965^32\}, \{10\ 7^3632\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4+1)(q^8-q^7+2q^6-q^5$
$+ q^4-q^3+2q^2-q+1)$

$[999]$ $\{10\ 9^263^22\}$
$+ q^{11}(q^2+q+1)^3(q^2-q+1)^2(q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5$
$+ 6q^4+3q^3+4q^2+2q+1)$

$[1050]$ $\{12\ 75^43\}, \{97^45\}$
$+ q^9(q^2+q+1)(q^4+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)$

$[1053]_1$ $\{11\ 9853^3\}, \{9^37431\}$
$+ q^{11}(q^2+q+1)^4(q^2-q+1)(q^{10}+3q^8-q^7+4q^6-q^5$
$+ 4q^4-q^3+3q^2+1)$

$[1053]_2$ $\{10\ 8^2643^2\}, \{10\ 8^25^242\}, \{10\ 87^24^22\}, \{9^2864^22\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)$
$(q^{16}+3q^{14}-2q^{13}+9q^{12}-3q^{11}+10q^{10}-6q^9+15q^8-6q^7$
$+ 10q^6-3q^5+9q^4-2q^3+3q^2+1)$

$[1080]$ $\{10\ 7^25^24^2\}, \{8^27^25^22\}$
$+ q^7(q^2+q+1)^3(q^2-q+1)(q^4+1)^3(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[1134]$ $\{10\ 96^253^2\}, \{9^276^232\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)(q^4+1)$
$\times (q^{10}-q^9+4q^8-4q^7+6q^6-5q^5+6q^4-4q^3+4q^2-q+1)$

[1161] $\{9^2854^23\}, \{98^2743^2\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)$
$\times (q^{16}+4q^{14}+7q^{12}-q^{11}+9q^{10}-3q^9+9q^8-3q^7+9q^6$
$-q^5+7q^4+4q^2+1)$

[1215] $\{11\ 7^254^3\}, \{8^375^21\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)(q^4+q^3+q^2+q+1)$
$(q^{10}-q^9+2q^8-2q^7+2q^6-q^5+2q^4-2q^3+2q^2-q+1)$

[1260] $\{11^2\ 54^33\}, \{98^371^2\}$
$+ q^{11}(q^2+q+1)^2(q^2+1)(q^4+1)(q^4+q^3+q^2+q+1)$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)$

$[1296]_1$ $\{11\ 9654^23\}, \{98^27631\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)^2(q^2+1)^2(q^4+1)^2$

$[1296]_2$ $\{10\ 9764^22\}, \{10\ 8^26532\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)$
$(q^{14}-q^{13}+4q^{12}-6q^{11}+13q^{10}-14q^9+18q^8-14q^7$
$+18q^6-14q^5+13q^4-6q^3+4q^2-q+1)$

$[1350]_1$ $\{11\ 964^4\}, \{8^4631\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)(q^4+1)(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)^2$

$[1350]_2$ $\{10\ 6^44^2\}, \{8^26^42\}$
$+ q^7(q^2+q+1)^3(q^2-q+1)(q^4+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)^2$

[1377] $\{10\ 86^33^2\}, \{9^26^342\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)$
$(2q^{14}-q^{13}+2q^{12}-q^{11}+6q^{10}-5q^9+6q^8-q^7+6q^6$
$-5q^5+6q^4-q^3+2q^2-q+2)$

[1404] $\{11\ 86^2542\}, \{10\ 876^241\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4+1)(q^{10}+3q^8-q^7+4q^6$
$-q^5+4q^4-q^3+3q^2+1)$

[1440] $\{10\ 76^343\}, \{986^352\}$
$+ q^7(q^2+q+1)^2(q^2+1)^2(q^4+1)^3(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[1458]_1$ $\{11\ 8754^23\}, \{11\ 7^26542\}, \{10\ 8765^21\}, \{98^27541\}$
$+ q^9(q^2 + q + 1)^5(q^2 - q + 1)^2(q^2 + 1)$
$\times (q^8 - q^7 + 2q^6 - q^5 + q^4 - q^3 + 2q^2 - q + 1)$

$[1458]_2$ $\{11\ 7^25^32\}, \{10\ 7^35^21\}$
$+ q^9(q^2 + q + 1)^5(q^2 - q + 1)(q^4 + 1)$
$\times (q^8 - q^7 + 2q^6 - q^5 + q^4 - q^3 + 2q^2 - q + 1)$

$[1485]$ $\{10\ 974^4\}, \{8^4532\}$
$+ q^9(q^2 + q + 1)^3(q^4 + q^3 + q^2 + q + 1)$
$(q^{14} - q^{13} + 3q^{12} - 2q^{11} + 5q^{10} - 5q^9 + 7q^8 - 5q^7 + 7q^6$
$- 5q^5 + 5q^4 - 2q^3 + 3q^2 - q + 1)$

$[1620]_1$ $\{11\ 95^343\}, \{987^331\}$
$+ q^9(q^2 + q + 1)^4(q^2 - q + 1)(q^2 + 1)(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)$
$\times (q^4 + q^3 + q^2 + q + 1)$

$[1620]_2$ $\{10^2\ 943^3\}, \{9^3832^2\}$
$+ q^{11}(q^2 + q + 1)^4(q^2 - q + 1)(q^2 + 1)(q^4 + 1)(q^4 + q^3 + q^2 + q + 1)$

$[1620]_3$ $\{10\ 865^33\}, \{97^3642\}$
$+ q^7(q^2 + q + 1)^4(q^2 - q + 1)^2(q^4 + 1)^2(q^4 - q^3 + q^2 - q + 1)$
$\times (q^4 + q^3 + q^2 + q + 1)$

$[1620]_4$ $\{9875^33\}, \{97^3543\}$
$+ q^7(q^2 + q + 1)^3(q^2 + 1)(q^4 + 1)(q^4 - q^3 + q^2 - q + 1)$
$\times (q^4 + q^3 + q^2 + q + 1)(q^8 - q^7 + 2q^6 - q^5 + q^4 - q^3 + 2q^2 - q + 1)$

$[1620]_5$ $\{9875^24^2\}, \{8^27^2543\}$
$+ q^7(q^2 + q + 1)^3(q^2 + 1)(q^4 + 1)(q^4 + q^3 + q^2 + q + 1)$
$\times (q^{10} - q^9 + 2q^8 - 2q^7 + 2q^6 - q^5 + 2q^4 - 2q^3 + 2q^2 - q + 1)$

$[1656]$ $\{11\ 7^2653^2\}, \{9^2765^21\}$
$+ q^9(q^2 + q + 1)^2(q^2 + 1)(q^4 + 1)(q^{14} + q^{13} + 4q^{12} + q^{11}$
$+ 7q^{10} + q^9 + 8q^8 + 8q^6 + q^5 + 7q^4 + q^3 + 4q^2 + q + 1)$

$[1719]$ $\{10\ 97643^2\}, \{9^286532\}$
$+ q^9(q^2 + q + 1)^2$
$(q^{20} + 6q^{18} - q^{17} + 17q^{16} - 5q^{15} + 34q^{14} - 12q^{13}$
$+ 47q^{12} - 18q^{11} + 53q^{10} - 18q^9 + 47q^8 - 12q^7 + 34q^6 - 5q^5$
$+ 17q^4 - q^3 + 6q^2 + 1)$

[1800] $\{12\ 765^2 43\}, \{11\ 76^3 51\}, \{987^2 65\}$
$+ q^9(q^2+q+1)^2(q^2+1)^2(q^4+1)(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)^2$

[1890] $\{11\ 6^5 1\}$
$+ q^9(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4-q^2+1)(q^4+q^3+q^2+q+1)$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)$

$[1944]_1$ $\{11\ 875^2 3^2\}, \{9^2 7^2 541\}$
$+ q^9(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4+1)(q^8+2q^6-2q^5$
$+ 4q^4-2q^3+2q^2+1)$

$[1944]_2$ $\{10\ 975^2 42\}, \{10\ 87^2 532\}$
$+ q^9(q^2+q+1)^5(q^2-q+1)^2(q^{10}-q^9+5q^8-5q^7+6q^6$
$- 4q^5+6q^4-5q^3+5q^2-q+1)$

$[2025]_1$ $\{10\ 85^4 4\}, \{87^4 42\}$
$+ q^7(q^2+q+1)^4(q^2-q+1)^2(q^4-q^2+1)(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)^2$

$[2025]_2$ $\{9^2 65^2 4^2\}, \{8^2 7^2 63^2\}$
$+ q^7(q^2+q+1)^3(q^2-q+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$
$\times (q^8-q^5+3q^4-q^3+1)$

[2187] $\{10^2\ 654^2 3\}, \{98^2 762^2\}$
$+ q^9(q^2+q+1)^6(q^2-q+1)^2(q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$

[2250] $\{12\ 85^2 4^3\}, \{12\ 6^3 5^2 2\}, \{10\ 7^2 6^3\}, \{8^3 7^2 4\}$
$+ q^9(q^2+q+1)^2(q^4+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^3$

[2430] $\{10\ 86^2 4^3\}, \{8^3 6^2 42\}$
$+ q^7(q^2+q+1)^4(q^2-q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)$
$\times (q^8-q^5+3q^4-q^3+1)$

$[2592]_1$ $\{10\ 98543^2\}, \{9^2 87432\}$
$+ q^9(q^2+q+1)^3(q^2+1)^3(q^4+1)(q^8+2q^6-2q^5+4q^4-2q^3+2q^2+1)$

$[2592]_2$ $\{10\ 7^2 6543\}, \{98765^2 2\}$
$+ q^7(q^2+q+1)^3(q^2+1)^3(q^4+1)^2(q^8-q^7+2q^6-q^5$
$+ q^4-q^3+2q^2-q+1)$

[2844] $\{10^2\ 74^3 3\}, \{98^3 52^2\}$
$+ q^9(q^2+q+1)^2(q^2+1)(q^4+1)$
$(q^{14}+2q^{13}+4q^{12}+4q^{11}+8q^{10}+7q^9+10q^8+7q^7+10q^6$
$+ 7q^5+8q^4+4q^3+4q^2+2q+1)$

[2880] $\{98764^3\}, \{8^36543\}$
$+ q^7(q^2+q+1)^2(q^2+1)^3(q^4+1)^2(q^4+q^3+q^2+q+1)$
$\times (q^6+q^4-2q^3+q^2+1)$
[2916] $\{10\ 86^2543\}, \{9876^242\}$
$+ q^7(q^2+q+1)^5(q^2-q+1)^2(q^2+1)(q^4+1)(q^8-q^7+2q^6-q^5+q^4$
$-q^3+2q^2-q+1)$
[3150] $\{11\ 10\ 54^4\}, \{8^4721\}$
$+ q^9(q^2+q+1)^2(q^2+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)$
[3204] $\{10\ 7^265^22\}$
$+ q^7(q^2+q+1)^2(q^4+1)^2(q^{16}+q^{15}+4q^{14}+2q^{13}$
$+9q^{12}+3q^{11}+13q^{10}+4q^9$
$+15q^8+4q^7+13q^6+3q^5+9q^4+2q^3+4q^2+q+1)$
[3375] $\{11\ 765^24^2\}, \{10\ 6^52\}, \{8^27^2651\}$
$+ q^7(q^2+q+1)^3(q^2-q+1)(q^4-q^3+q^2-q+1)^2$
$\times (q^4+q^3+q^2+q+1)^3$
[3492] $\{9^26^2543\}, \{9876^23^2\}$
$+ q^7(q^2+q+1)^2(q^2+1)(q^4+1)(q^{18}+q^{17}+4q^{16}+2q^{15}$
$+8q^{14}+2q^{13}+12q^{12}+2q^{11}+15q^{10}+3q^9+15q^8$
$+2q^7+12q^6+2q^5+8q^4+2q^3+4q^2+q+1)$
[3600] $\{11\ 765^33\}, \{97^3651\}$
$+ q^7(q^2+q+1)^2(q^2+1)^2(q^4+1)^2(q^4-q^3+q^2-q+1)^2$
$(q^4+q^3+q^2+q+1)^2$
[4050] $\{8^26^25^24\}, \{87^26^24^2\}$
$+ q^5(q^2+q+1)^4(q^2-q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)^2$
$\times (q^4-q^3+q^2-q+1)(q^4-q^2+1)$
[4140] $\{10\ 76^352\}$
$+ q^7(q^2+q+1)^2(q^4+1)(q^{14}+3q^{12}+4q^{10}-q^9+5q^8-q^7$
$+5q^6-q^5+4q^4+3q^2+1)$
[4374] $\{9^363^3\}$
$+ q^9(q^2+q+1)^4(q^2+1)$
$(q^{14}+q^{13}+q^{12}+q^{11}+5q^{10}-q^9+4q^8+3q^7+4q^6-q^5$
$+5q^4+q^3+q^2+q+1)$
[4410] $\{7^365^3\}$

$+ q^5(q^2+q+1)^2(q^2+1)(q^4+q^3+q^2+q+1)(q^4-q^2+1)$
$(q^6+q^5+q^4+q^3+q^2+q+1)^2(q^6-q^5+q^4-q^3+q^2-q+1)$

[4500] $\{976^34^2\}, \{8^26^353\}$
$+ q^5(q^2+q+1)^2(q^4+q^3+q^2+q+1)^3(q^4-q^3+q^2-q+1)^2$
$\times (q^4+1)^2$

[4725] $\{12\ 65^44\}, \{87^46\}$
$+ q^7(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)(q^6+q^3+1)$

[4860] $\{11\ 6^35^23\}, \{97^26^31\}$
$+ q^7(q^2+q+1)^5(q^2-q+1)^2(q^4+q^3+q^2+q+1)(q^4-q^2+1)(q^4+1)$

[4968] $\{10\ 875^243\}, \{987^2542\}$
$+ q^7(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4+1)$
$(q^{14}+q^{13}+4q^{12}+q^{11}+7q^{10}+q^9+8q^8+8q^6+q^5$
$+ 7q^4+q^3+4q^2+q+1)$

[5400] $\{11\ 85^34^2\}, \{8^27^341\}$
$+ q^7(q^2+q+1)^3(q^2+1)(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)^2$
$\times (q^4+1)^2$

[6048] $\{9^2764^23\}, \{98^2653^2\}$
$+ q^7(q^2+q+1)^3(q^2-q+1)^2(q^2+1)(q^4+1)$
$\times (q^{12}+3q^{11}+6q^{10}+5q^9+6q^8+4q^7+6q^6+4q^5$
$+ 6q^4+5q^3+6q^2+3q+1)$

[6300] $\{96^53\}$
$+ q^5(q^2+q+1)^2(q^2+1)(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)$
$\times (q^4+1)(q^4-q^2+1)(q^6+q^5+q^4+q^3+q^2+q+1)$

[6480] $\{10\ 9654^3\}, \{8^37632\}$
$+ q^7(q^2+q+1)^3(q^2+1)^3(q^4+q^3+q^2+q+1)(q^4+1)$
$\times (q^8-q^5+3q^4-q^3+1)$

[6750] $\{87^265^24\}$
$+ q^5(q^2+q+1)^2(q^4+q^3+q^2+q+1)^3(q^{12}-q^{11}+3q^{10}-2q^9+2q^8$
$- q^7+2q^6-q^5+2q^4-2q^3+3q^2-1+1)$

[7875] $\{9765^4\}, \{7^4653\}$
$+ q^5(q^2+q+1)^2(q^4+q^3+q^2+q+1)^3(q^4-q^3+q^2-q+1)$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)(q^6-q^5+q^4-q^3+q^2-q+1)$

[8625] $\{976^353\}$
$$+ q^5(q^2+q+1)(q^4-q^3+q^2-q+1)(q^{14}+3q^{12}+4q^{10}-q^9 + 5q^8-q^5+4q^4+3q^2+1)$$

[10125] $\{10\ 6^35^24\}, \{87^26^32\}$
$$+ q^5(q^2+q+1)^4(q^2-q+1)^2(q^4+q^3+q^2+q+1)^3 \times (q^4-q^3+q^2-q+1)(q^4-q^2+1)$$

[11940] $\{97^265^23\}$
$$+ q^5(q^2+q+1)(q^2+1)(q^4+1)(q^4+q^3+q^2+q+1) (q^{20}+2q^{19}+5q^{18}+4q^{17}+10q^{16}+8q^{15}+17q^{14} + 10q^{13}+20q^{12}+11q^{11}+23q^{10}+11q^9+20q^8+10q^7 + 17q^6+8q^5+10q^4+4q^3+5q^2+2q+1)$$

[12150] $\{986^254^2\}, \{8^276^243\}$
$$+ q^5(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4+q^3+q^2+q+1) \times (q^4-q^3+q^2-q+1)(q^8-q^5+3q^4-q^3+1)$$

[12 960] $\{10\ 765^34\}, \{87^3652\}$
$$+ q^5(q^2+q+1)^4(q^2-q+1)(q^2+1)^3(q^4+q^3+q^2+q+1)(q^4+1)^2 \times (q^4-q^2+1)$$

[13230] $\{7^26^35^2\}$
$$+ q^3(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^4+1)(q^6+q^5+q^4 + q^3+q^2+q+1)^2(q^6+q^3+1)(q^6-q^5+q^4-q^3+q^2-q+1)$$

[14175] $\{86^54\}$
$$+ q^3(q^2+q+1)^3(q^2-q+1)(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1) (q^4-q^2+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^6+q^3+1)$$

[17010] $\{11\ 65^5\}, \{7^561\}$
$$+ q^5(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4+q^3+q^2+q+1)(q^4-q^2+1) \times (q^6+q^5+q^4+q^3+q^2+q+1)(q^6+q^3+1)$$

[25200] $\{876^354\}$
$$+ q^3(q^2+q+1)^2(q^2+1)^3(q^4+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1) \times (q^4+q^3+q^2+q+1)^2(q^6+q^5+q^4+q^3+q^2+q+1)$$

[31500] $\{96^35^3\}, \{7^36^33\}$
$$+ q^3(q^2+q+1)^2(q^2+1)(q^4+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1) \times (q^4+q^3+q^2+q+1)^3(q^6+q^5+q^4+q^3+q^2+q+1)$$

[62370] $\{76^5 5\}$

$q(q^2+q+1)^3(q^2+1)(q^4-q^2+1)(q^6+q^3+1)(q^{10}+q^9+q^8+q^7$
$+q^6+q^5+q^4+q^3+q^2+q+1)$

[−3] $\{12\ 10\ 8641^2\}, \{12\ 10\ 863^2\}, \{12\ 10\ 85^2 2\}, \{12\ 10\ 7^2 42\}, \{12\ 9^2 642\}$
$\{11^2\ 8642\}$
$-q^{20}(q^2+q+1)$

[−6] $\{12\ 8^2 6^2 2\}, \{12\ 10\ 84^2 2^2\}, \{12\ 10\ 6^2 4^2\}, \{10^2\ 8^2 42\}$
$-q^{18}(q^2+q+1)(q^4+1)$

[−9] $\{12\ 10\ 853^2 1\}, \{12\ 10\ 75^2 3\}, \{12\ 97^2 52\}, \{11\ 9^2 742\}$
$-q^{18}(q^2+q+1)^2(q^2-q+1)$

[−12] $\{12\ 10\ 85421\}, \{12\ 10\ 7643\}, \{12\ 98652\}, \{11\ 10\ 8742\}$
$-q^{18}(q^2+q+1)(q^2+1)^2$

[−15] $\{12\ 10\ 862^3\}, \{12\ 10\ 84^3\}, \{12\ 10\ 6^3 2\}, \{12\ 8^3 42\}\{10^3 642\}$
$-q^{18}(q^2+q+1)(q^4+q^3+q^2+q+1)$

$[-18]_1$ $\{12\ 10\ 7^2 321\}, \{12\ 10\ 7651^2\}, \{12\ 9^2 6321\}, \{12\ 9^2 543\}$
$\{12\ 98741^2\}, \{12\ 9873^2\}, \{11^2\ 8543\}, \{11\ 10\ 963^2\}, \{11^2\ 7652\}$
$\{11^2\ 86321\}, \{11\ 10\ 9641^2\}, \{11\ 10\ 95^2 2\}, \{11^2\ 86321\}$
$-q^{18}(q^2+q+1)^2(q^2+1)$

$[-18]_2$ $\{12\ 10\ 6^2 3^2 2\}, \{12\ 8^2 5^2 4\}, \{12\ 87^2 4^2\}, \{10^2\ 7^2 62\}, \{10\ 9^2 6^2 2\}$
$-q^{16}(q^2+q+1)^2(q^2-q+1)(q^4+1)$

$[-27]_1$ $\{12\ 9^2 5^2 1^2\}, \{11^2\ 7^2 3^2\}, \{11^2\ 85^2 1^2\}, \{11^2\ 7^2 41^2\}$
$-q^{18}(q^2+q+1)^3$

$[-27]_2$ $\{12\ 10\ 7542^2\}, \{12\ 10\ 6^2 431\}, \{12\ 97^2 3^2 1\}, \{12\ 9764^2\}$
$\{12\ 8^2 653\}, \{11\ 9^2 5^2 3\}\{11\ 986^2 2\}, \{10^2\ 8752\}$
$-q^{16}(q^2+q+1)^3(q^2-q+1)^2$

$[-36]_1$ $\{12\ 10\ 75431\}, \{12\ 9863^2 1\}, \{12\ 985^2 21\}, \{12\ 97^2 421\}$
$\{12\ 97653\}, \{11\ 10\ 85^2 3\}\{11\ 10\ 7^2 43\}, \{11\ 9^2 643\}, \{11\ 98752\}$
$-q^{16}(q^2+q+1)^2(q^2-q+1)(q^2+1)^2$

$[-36]_2$ $\{12\ 10\ 74^3 1\}, \{12\ 96^3 3\}, \{11\ 10\ 94^2 2^2\}, \{11\ 8^3 52\}, \{10^2\ 8^2 321\}$
$-q^{16}(q^2+q+1)^2(q^2+1)(q^4+1)$

$[-36]_3$ $\{12\ 8^2 4^3 2\}, \{12\ 86^3 2^2\}, \{10^2\ 6^3 4\}, \{10\ 8^3 4^2\}$
$-q^{14}(q^2+q+1)^2(q^2-q+1)(q^4+1)^2$

$[-45]$ $\{12\ 10\ 843^22\}, \{12\ 10\ 65^24\}, \{12\ 87^262\}, \{10\ 9^2842\}$
$-\ q^{16}(q^2+q+1)^2(q^2-q+1)(q^4+q^3+q^2+q+1)$

$[-48]$ $\{12\ 986421\}, \{11\ 10\ 8643\}$
$-\ q^{16}(q^2+q+1)(q^2+1)^4$

$[-54]_1$ $\{12\ 10\ 7632^2\}, \{12\ 10\ 6^2521\}, \{12\ 9854^2\}, \{12\ 8^2743\}$
$\{11^2\ 763^21\}, \{11\ 10\ 76^22\}, \{11\ 9^2651^2\}\{11^2\ 75^23\}$
$\{11\ 10\ 7^251^2\}, \{11\ 10\ 953^21\}, \{11\ 9^27321\}, \{10^2\ 9652\}$
$-\ q^{16}(q^2+q+1)^3(q^2-q+1)(q^2+1)$

$[-54]_2$ $\{12\ 87^23^22\}, \{10\ 9^25^24\}$
$-\ q^{14}(q^2+q+1)^3(q^2-q+1)^2(q^4+1)$

$[-54]_3$ $\{11\ 87^23^3\}, \{9^35^241\}$
$-\ q^{12}(q^2+q+1)^3(q^2+1)(q^{10}-2q^9+2q^8-2q^7+3q^6$
$-\ 3q^5+3q^4-2q^3+2q^2-2q+1)$

$[-69]$ $\{12\ 8^2642^2\}, \{10^2\ 864^2\}$
$-\ q^{14}(q^2+q+1)(q^(12)+2q^(10)+6q^8+5q^6+2q^2+1)$

$[-72]$ $\{11^2\ 76421\}, \{11\ 10\ 95421\}, \{11\ 10\ 87321\}, \{11\ 10\ 8651^2\}$
$-\ q^{16}(q^2+q+1)^2(q^2+1)^3$

$[-75]$ $\{12\ 10\ 5^243^2\}, \{12\ 7^265^2\}, \{9^287^22\}$
$-\ q^{14}(q^2+q+1)(q^4+q^3+q^2+q+1)^2(q^4-q^3+q^2-q+1)$

$[-81]_1$ $\{12\ 9^2532^2\}, \{12\ 9^24^231\}, \{12\ 976^21^2\}, \{12\ 8^2751^2\}$
$\{11^2\ 8532^2\}, \{11^2\ 84^231\}, \{11^2\ 754^2\}, \{11^2\ 6^253\}$
$\{11\ 9^2542^2\}, \{11\ 98^23^2\}, \{11\ 98^241^2\}, \{10^2\ 873^21\}$
$\{10^2\ 9741^2\}, \{10^2\ 973^2\}$
$-\ q^{16}(q^2+q+1)^4(q^2-q+1)$

$[-81]_2$ $\{12\ 9763^22\}, \{12\ 9754^21\}, \{12\ 96^2531\}, \{12\ 87^2431\}$
$\{11\ 985^24\}, \{11\ 976^23\}, \{11\ 8^2753\}, \{10\ 9^2653\}$
$-\ q^{14}(q^2+q+1)^4(q^2-q+1)^3$

$[-81]_3$ $\{12\ 96^23^3\}, \{12\ 7^34^21\}, \{11\ 8^25^3\}, \{9^36^23\}$
$-\ q^{14}(q^2+q+1)^3(q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$

$[-90]_1$ $\{12\ 10\ 64^32\}, \{10\ 8^362\}, \{12\ 86^34\}$
$-\ q^{14}(q^2+q+1)^2(q^2-q+1)(q^4+1)(q^4+q^3+q^2+q+1)$

$[-90]_2$ $\{12\ 9872^3\}, \{10^3\ 543\}, \{12\ 8^3321\}, \{11\ 10\ 94^3\}$
$\{10^3\ 6321\}, \{11\ 10\ 962^3\}$
$-\ q^{16}(q^2+q+1)^2(q^2+1)(q^4+q^3+q^2+q+1)$

$[-90]_3$ $\{12\ 7^3 3^3\}, \{9^3 5^3\}$
$- q^{14}(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^6+q^4-2q^3+q^2+1)$

$[-108]_1$ $\{12\ 86^2 43^2\}, \{9^2 86^2 4\}, \{12\ 7^2 64^2 2\}, \{10\ 8^2 65^2\}$
$\{10\ 9^2 4^3 2\}, \{10\ 8^3 3^2 2\}$
$- q^{12}(q^2+q+1)^3(q^2-q+1)^2(q^4+1)^2$

$[-108]_2$ $\{11\ 10\ 8542^2\}, \{10^2\ 87421\}, \{11\ 9873^2 1\}, \{11\ 9873^2 1\}$
$- q^{14}(q^2+q+1)^3(q^2-q+1)^2(q^2+1)^2$

$[-108]_3$ $\{11\ 9^2 4^3 1\}, \{11\ 8^3 3^2 1\}$
$- q^{14}(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4+1)$

$[-117]$ $\{12\ 96^2 52^2\}, \{10^2\ 76^2 3\}, \{12\ 96^2 4^2 1\}, \{11\ 8^2 6^2 3\}$
$\{12\ 8^2 54^2 1\}, \{118^2 74^2\}$
$- q^{14}(q^2+q+1)^2(q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)$

$[-135]_1$ $\{11^2\ 7^2 2^3\}, \{10^3\ 5^2 1^2\}, \{11^2\ 6^3 1^2\}$
$- q^{16}(q^2+q+1)^3(q^4+q^3+q^2+q+1)$

$[-135]_2$ $\{11^2\ 653^3\}, \{9^3 761^2\}, \{11^2\ 5^3 32\}, \{10\ 97^3 1^2\}$
$- q^{14}(q^2+q+1)^3(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)$

$[-144]_1$ $\{12\ 10\ 65432\}, \{10\ 98762\}, \{12\ 984^2 32\}, \{10\ 98^2 43\}$
$\{12\ 876^2 21\}, \{11\ 10\ 6^2 54\}, \{11\ 10\ 84^3 1\}, \{11\ 8^3 421\}$
$\{11\ 10\ 6^3 21\}, \{12\ 87654\}$
$- q^{14}(q^2+q+1)^2(q^2+1)^3(q^4+1)$

$[-144]_2$ $\{11\ 10\ 85431\}, \{11\ 987421\}, \{11\ 10\ 76521\}$
$- q^{14}(q^2+q+1)^2(q^2-q+1)(q^2+1)^4$

$[-153]$ $\{12\ 8^2 5^2 2^2\}, \{10^2\ 7^2 4^2\}$
$- q^{14}(q^2+q+1)^2(q^2-q+1)(q^8+2q^7+4q^6+q^5+q^4+q^3+4q^2+2q+1)$

$[-162]_1$ $\{12\ 9853^2 2\}, \{10\ 9^2 743\}, \{12\ 87^2 521\}, \{11\ 10\ 75^2 4\}$
$\{11\ 10\ 7^2 32^2\}, \{10^2\ 95^2 21\}, \{11\ 9^2 632^2\}, \{10^2\ 963^2 1\}$
$- q^{14}(q^2+q+1)^4(q^2-q+1)^2(q^2+1)$

$[-162]_2$ $\{12\ 97642^2\}, \{10^2\ 8653\}, \{12\ 975^2 31\}, \{11\ 97^2 53\}, \{12\ 8^2 6431\}, \{11\ 9864^2\}$
$- q^{14}(q^2+q+1)^3(q^2-q+1)(q^6+3q^4-2q^3+3q^2+1)$

$[-162]_3$ $\{12\ 8754^2 2\}, \{10\ 8^2 754\}, \{12\ 86^2 532\}, \{10\ 976^2 4\}, \{11\ 984^3 2\}, \{10\ 8^3 431\}$
$- q^{12}(q^2+q+1)^4(q^2-q+1)^3(q^4+1)$

$[-162]_4$ $\{11\ 10\ 75^2 2^2\}, \{10^2\ 7^2 521\}$
$- q^{14}(q^2+q+1)^4(q^2-q+1)(2q^2-3q+2)$

$[-162]_5$ $\{11\ 87^24^21\}, \{11\ 87^24^21\}$

$-q^{12}(q^2+q+1)^3(q^2+1)(q^{10}-2q^9+4q^8-5q^7+6q^6-5q^5$
$+6q^4-5q^3+4q^2-2q+1)$

$[-162]_6$ $\{11\ 976531\}$

$-q^{12}(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^8-2q^7+4q^6-5q^5$
$+7q^4-5q^3+4q^2-2q+1)$

$[-180]_1$ $\{12\ 10\ 743^3\}, \{9^3852\}, \{12\ 10\ 5^341\}, \{11\ 87^32\}$
$\{12\ 965^3\}, \{12\ 7^363\}$

$-q^{14}(q^2+q+1)^2(q^2+1)(q^4+1)(q^4+q^3+q^2+q+1)$

$[-180]_2$ $\{12\ 874^33\}, \{98^354\}, \{12\ 76^332\}, \{10\ 96^35\}$
$\{11\ 10\ 5^33^2\}, \{9^27^321\}$

$-q^{12}(q^2+q+1)^2(q^2+1)(q^4+1)(q^4-q^3+2q^2-q+1)$
$\times(q^4+q^3+q^2+q+1)$

$[-198]_1$ $\{11\ 10\ 76431\}, \{11\ 986521\}$

$-q^{14}(q^2+q+1)^2(q^2-q+1)(q^2+1)(2q^6+4q^4-q^3+4q^2+2)$

$[-198]_2$ $\{11\ 8^2652^2\}, \{10^2764^21\}, \{11\ 8^264^21\}, \{10^2\ 7652^2\}$

$-q^{12}(q^2+q+1)^2(q^2+1)(q^{12}-q^{11}+3q^{10}-3q^9+7q^8-5q^7$
$+7q^6-5q^5+7q^4-3q^3+3q^2-q+1)$

$[-201]$ $\{12\ 7^2643^2\}, \{9^2865^2\}$

$-q^{12}(q^2+q+1)(q^{16}+q^{15}+4q^{14}+2q^{13}+6q^{12}+3q^{11}$
$+9q^{10}+3q^9+9q^8+3q^7+9q^6+3q^5+6q^4+2q^3+4q^2+q+1)$

$[-207]$ $\{10^2\ 863^22\}, \{10\ 9^2642^2\}$

$-q^{12}(q^2+q+1)^2(q^2-q+1)(q^{12}+2q^{10}+6q^8+5q^6+6q^4+2q^2+1)$

$[-216]_1$ $\{11^2\ 74^232\}, \{10\ 98^251^2\}, \{11^2\ 654^21\}, \{11\ 8^2761^2\}$

$-q^{14}(q^2+q+1)^3(q^2+1)^2(q^4+1)$

$[-216]_2$ $\{11\ 10\ 8632^2\}, \{10^2\ 96421\}$

$-q^{14}(q^2+q+1)^3(q^2-q+1)(q^2+1)^3$

$[-216]_3$ $\{11\ 10\ 65^232\}, \{10\ 97^2621\}$

$-q^{12}(q^2+q+1)^3(q^2-q+1)^2(q^2+1)^2(q^4+1)$

$[-243]$ $\{12\ 8^2732^2\}, \{10^2\ 954^2\}, \{11^2\ 753^22\}, \{10\ 9^2751^2\}$
$\{11^2\ 65^231\}, \{11\ 97^261^2\}$

$-q^{14}(q^2+q+1)^5(q^2-q+1)^2$

$[-270]$ $\{11\ 10\ 943^22\}, \{10\ 9^28321\}$

$-q^{14}(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4+q^3+q^2+q+1)$

$[-288]$ $\{12\ 96543^2\}, \{9^28763\}, \{12\ 7^26541\}, \{11\ 8765^2\}$
$- q^{12}(q^2+q+1)^2(q^2+1)^3(q^4+1)^2$

$[-315]$ $\{12\ 9^23^4\}, \{9^43^2\}, \{12\ 7^41^2\}, \{11^2\ 5^4\}, \{11^2\ 83^4\}$
$\{9^441^2\}$
$- q^{14}(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$

$[-324]_1$ $\{12\ 9654^22\}, \{10\ 8^2763\}, \{12\ 86^2541\}, \{11\ 876^24\}$
$\{11\ 10\ 753^3\}, \{9^37521\}, \{11\ 10\ 74^32\}, \{10\ 8^3521\}$
$\{11\ 97652^2\}, \{10^2\ 76531\}$
$- q^{12}(q^2+q+1)^4(q^2-q+1)^2(q^2+1)(q^4+1)$

$[-324]_2$ $\{11\ 10\ 75432\}, \{10\ 987521\}$
$- q^{12}(q^2+q+1)^4(q^2-q+1)^3(q^2+1)^2$

$[-324]_3$ $\{11\ 9764^21\}, \{11\ 8^26531\}$
$- q^{12}(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^6-2q^5+5q^2-2q+1)$

$[-324]_4$ $\{9^275^243\}, \{987^253^2\}$
$- q^8(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4+1)(q^{10}-2q^9+2q^8-2q^7$
$+ 3q^6-3q^5+3q^4-2q^3+2q^2-2q+1)$

$[-333]$ $\{11^2\ 6^242^2\}, \{10^2\ 86^21^2\}$
$- q^{14}(q^2+q+1)^2(q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5$
$+ 6q^4+3q^3+4q^2+2q+1)$

$[-351]$ $\{12\ 875^232\}, \{10\ 97^254\}, \{11\ 96^331\}, \{11\ 8^273^22\}, \{10\ 9^254^21\}$
$- q^{12}(q^2+q+1)^3(q^2-q+1)(q^{10}+3q^8-q^7+4q^6-q^5+4q^4$
$- q^3+3q^2+1)$

$[-360]$ $\{10^2\ 5^343\}, \{987^32^2\}$
$- q^{10}(q^2+q+1)^2(q^2+1)(q^4+1)^2(q^4-q^3+2q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[-387]$ $\{11\ 8^2742^2\}, \{10^2\ 854^21\}$
$- q^{12}(q^2+q+1)^2(q^{14}+4q^{12}-q^{11}+9q^{10}-3q^9+13q^8$
$- 3q^7+13q^6-3q^5+9q^4-q^3+4q^2+1)$

$[-405]_1$ $\{12\ 95^342\}, \{10\ 87^33\}, \{12\ 865^31\}, \{11\ 7^364\}$
$- q^{12}(q^2+q+1)^4(q^2-q+1)(q^4-q^3+2q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[-405]_2$ $\{11\ 98^22^3\}, \{10^3\ 4^231\}, \{10^3\ 532^2\}, \{10^2\ 972^3\}$
$- q^{14}(q^2+q+1)^4(q^2-q+1)(q^4+q^3+q^2+q+1)$

$[-405]_3$ $\{11\ 76^33^2\}, \{9^26^351\}$
$- q^{10}(q^2+q+1)^3(q^4-q^3+2q^2-q+1)(q^4+q^3+q^2+q+1)$
$\times (q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$

$[-420]$ $\{12\ 10\ 54^33\}, \{98^372\}, \{12\ 76^35\}$
$- q^{12}(q^2+q+1)(q^2+1)(q^4+1)(q^4+q^3+q^2+q+1)$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)$

$[-432]$ $\{11\ 10\ 65^241\}, \{11\ 87^2621\}$
$- q^{12}(q^2+q+1)^3(q^2-q+1)(q^2+1)^2(q^4+1)$

$[-450]$ $\{11\ 10\ 5^41\}, \{11\ 7^421\}$
$- q^{12}(q^2+q+1)^2(q^2+1)(q^4-q^3+2q^2-q+1)(q^4+q^3+q^2+q+1)^2$

$[-459]$ $\{10^2\ 7^23^22\}, \{10\ 9^25^22^2\}$
$- q^{12}(q^2+q+1)^3(q^2-q+1)^2(q^8+2q^7+4q^6+q^5+q^4$
$+ q^3+4q^2+2q+1)$

$[-468]$ $\{12\ 876432\}, \{10\ 98654\}, \{11\ 10\ 6^2432\}, \{10\ 986^221\}$
$- q^{12}(q^2+q+1)^2(q^2+1)^2$
$\times (q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)$

$[-486]_1$ $\{12\ 974^23^2\}, \{9^28^253\}, \{12\ 7^26^231\}, \{11\ 96^25^2\}$
$- q^{12}(q^2+q+1)^5(q^2-q+1)^2(q^4+1)$

$[-486]_2$ $\{11\ 9863^22\}, \{10\ 9^26431\}$
$- q^{12}(q^2+q+1)^4(q^2-q+1)^2(q^6+3q^4-2q^3+3q^2+1)$

$[-486]_3$ $\{11\ 9763^3\}, \{9^36531\}, \{11\ 876^22^2\}, \{10^26^2541\}$
$- q^{12}(q^2+q+1)^4(q^2-q+1)(q^2+1)$
$\times (2q^6-2q^5+2q^4-q^3+2q^2-2q+2)$

$[-486]_4$ $\{11\ 975^241\}, \{11\ 87^2531\}$
$- q^{12}(q^2+q+1)^5(q^2-q+1)^2(q^2+1)(2q^2-3q+2)$

$[-486]_5$ $\{11\ 965^242\}, \{10\ 87^2631\}, \{11\ 876^22^2\}, \{10^2\ 6^2541\}$
$- q^{10}(q^2+q+1)^5(q^2-q+1)^4(q^4+1)$

$[-486]_6$ $\{11\ 8764^22\}, \{10\ 8^26541\}$
$- q^{10}(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^{10}-2q^9+4q^8-5q^7+6q^6$
$- 5q^5+6q^4-5q^3+4q^2-2q+1)$

$[-486]_7$ $\{10\ 8764^23\}, \{98^26542\}$
$- q^8((q^2+q+1)^5(q^2-q+1)^2(q^2+1)(q^{10}-2q^9+2q^8-2q^7+3q^6$
$- 3q^5+3q^4-2q^3+2q^2-2q+1)$

$[-540]$ $\{11\ 9^2 4 3^3\}, \{9^3 8 3^2 1\}$

$$-q^{12}(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4+1)(q^4+q^3+q^2+q+1)$$

$[-549]$ $\{10^2\ 6^3 2^2\}$

$$-q^{12}(q^2+q+1)^2(q^2-q+1)(q^{12}+3q^{11}+7q^{10}+6q^9+6q^8 + 4q^7+7q^6+4q^5+6q^4+6q^3+7q^2+3q+1)$$

$[-558]$ $\{11\ 8764 3^2\}, \{9^2 86541\}$

$$-q^{10}(q^2+q+1)^2(q^2+1)(q^{16}-q^{15}+4q^{14}-3q^{13}+9q^{12} - 6q^{11}+13q^{10} - 9q^9+15q^8-9q^7+13q^6-6q^5+9q^4-3q^3+4q^2-q+1)$$

$[-603]$ $\{10\ 8^2 6 4^2 2\}$

$$-q^{10}(q^2+q+1)^2(q^2-q+1)(2q^{16}+4q^{14}-q^{13}+12q^{12}-2q^{11} + 13q^{10}-4q^9+19q^8-4q^7+13q^6-2q^5+12q^4-q^3+4q^2+2)$$

$[-630]$ $\{98 5^5\}, \{7^5 43\}$

$$-q^8(q^2+q+1)^2(q^2+1)(q^4-q^2+1)(q^4+q^3+q^2+q+1) \times (q^6-q^5+q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$$

$[-648]_1$ $\{11\ 985432\}, \{10\ 987431\}$

$$-q^{12}(q^2+q+1)^4(q^2-q+1)(q^2+1)^3(2q^2-3q+2)$$

$[-648]_2$ $\{11\ 9 7^2 4 2^2\}, \{10^2\ 8 5^2 31\}$

$$-q^{12}(q^2+q+1)^4(q^2-q+1)^2(q^6+q^5+3q^4-2q^3+3q^2+q+1)$$

$[-648]_3$ $\{11\ 97 4^3 3\}, \{9 8^3 531\}, \{11\ 8 6^2 5^2 1\}, \{11\ 7^2 6^2 41\}$

$$-q^{10}(q^2+q+1)^4(q^2-q+1)^2(q^2+1)(q^4+1)^2$$

$[-648]_4$ $\{11\ 876532\}, \{10\ 976541\}$

$$-q^{10}(q^2+q+1)^3(q^2+1)^3(q^{10}-2q^9+4q^8-5q^7+6q^6-5q^5 + 6q^4-5q^3+4q^2-2q+1)$$

$[-648]_5$ $\{10^2\ 9 4^2 32\}, \{10\ 9 8^2 3 2^2\}$

$$-q^{12}(q^2+q+1)^4(q^2-q+1)(q^2+1)^2(q^4+1)$$

$[-648]_6$ $\{10\ 98 4^3 3\}, \{9 8^3 432\}$

$$-q^{10}(q^2+q+1)^3(q^2+1)^2(q^4+1) \times (q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$$

$[-648]_7$ $\{10\ 8 6^2 5^2 2\}, \{10 7^2 6^2 42\}$

$$-q^8(q^2+q+1)^4(q^2-q+1)^3(q^4+1)^3$$

$[-702]$ $\{11\ 96 5^2 3^2\}, \{9^2 7^2 631\}$

$$-q^{10}(q^2+q+1)^3(q^2-q+1)(q^4+1) \times (q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)$$

$[-708]$ $\{10^2\ 8642^2\}$

$$-q^{12}(q^2+q+1)(q^{16}+2q^{15}+8q^{14}+7q^{13}+18q^{12}+13q^{11}+33q^{10}+18q^9+36q^8+33q^6+13q^5+18q^4+7q^3+8q^2+2q+1)$$

$[-711]$ $\{12\ 8^253^3\}, \{9^374^2\}, \{12\ 7^352^2\}, \{10^2\ 75^3\}$

$$-q^{12}(q^2+q+1)^2(q^{14}+2q^{13}+4q^{12}+4q^{11}+8q^{10}+7q^9+10q^8+7q^7+10q^6+7q^5+8q^4+4q^3+4q^2+2q+1)$$

$[-720]_1$ $\{12\ 7654^3\}, \{8^3765\}, \{12\ 6^3543\}, \{9876^3\}$

$$-q^{10}(q^2+q+1)^2(q^2+1)^3(q^4+1)(q^4-q^2+1)(q^4+q^3+q^2+q+1)$$

$[-720]_2$ $\{11\ 10\ 843^3\}, \{9^38421\}$

$$-q^{12}(q^2+q+1)^2(q^2+1)^3(q^4+1)(q^4+q^3+q^2+q+1)$$

$[-729]_1$ $\{11\ 7^26^232\}, \{10\ 96^25^21\}$

$$-q^{10}(q^2+q+1)^5(q^2-q+1)^2 \times (q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$$

$[-729]_2$ $\{10^2\ 953^22\}, \{10\ 9^2732^2\}$

$$-q^{12}(q^2+q+1)^6(q^2-q+1)^3$$

$[-756]$ $\{10\ 96^332\}$

$$-q^{10}(q^2+q+1)^3(q^2+1)(q^4+1) \times (q^{10}-q^9+4q^8-4q^7+6q^6-5q^5+6q^4-4q^3+4q^2-q+1)$$

$[-792]$ $\{10\ 986432\}$

$$-q^{10}(q^2+q+1)^2(q^2+1)^3(q^{12}-q^{11}+3q^{10}-3q^9+7q^8-5q^7+7q^6-5q^5+7q^4-3q^3+3q^2-q+1)$$

$[-810]_1$ $\{11^2\ 64^23^2\}, \{9^28^261^2\}, \{11^2\ 5^24^22\}, \{10\ 8^27^21^2\}$

$$-q^{12}(q^2+q+1)^4(q^2-q+1)(q^4+1)(q^4+q^3+q^2+q+1)$$

$[-810]_2$ $\{11\ 95^42\}, \{10\ 7^431\}$

$$-q^{10}(q^2+q+1)^4(q^2-q+1)(q^4+1)(q^4-q^3+2q^2-q+1) \times (q^4+q^3+q^2+q+1)$$

$[-810]_3$ $\{11\ 874^4\}, \{8^4541\}$

$$-q^{10}(q^2+q+1)^3(q^2+1)(q^4+q^3+q^2+q+1) \times (q^{10}-q^9+2q^8-2q^7+2q^6-q^5+2q^4-2q^3+2q^2-q+1)$$

$[-810]_4$ $\{9^265^33\}, \{97^363^2\}$

$$-q^8(q^2+q+1)^3(q^4+1)(q^4-q^3+2q^2-q+1) \times (q^4+q^3+q^2+q+1)(q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$$

$[-846]$ $\{10^2\ 7543^2\}, \{9^2 8752^2\}$
$$-q^{10}(q^2+q+1)^2(q^4+1)$$
$$\times(q^{14}+4q^{12}+9q^{10}-2q^9+13q^8-3q^7+13q^6-2q^5+9q^4+4q^2+1)$$

$[-873]$ $\{9^3 543^2\}, \{9^2 873^3\}$
$$-q^{10}(q^2+q+1)^2(q^{18}+q^{17}+4q^{16}+2q^{15}+8q^{14}+2q^{13}$$
$$+12q^{12}+2q^{11}+15q^{10}+3q^9+15q^8+2q^7+12q^6+2q^5$$
$$+8q^4+2q^3+4q^2+q+1)$$

$[-891]$ $\{10\ 87^2 43^2\}, \{9^2 85^2 42\}$
$$-q^{10}(q^2+q+1)^4(q^2-q+1)$$
$$\times(2q^{12}-3q^{11}+6q^{10}-4q^9+7q^8-8q^7+11q^6-8q^5+7q^4$$
$$-4q^3+6q^2-3q+2)$$

$[-900]_1$ $\{12\ 85^3 43\}, \{987^3 4\}, \{12\ 765^3 2\}, \{10\ 7^3 65\}$
$$-q^{10}(q^2+q+1)^2(q^2+1)(q^4+1)(q^4-q^3+2q^2-q+1)$$
$$\times(q^4+q^3+q^2+q+1)^2$$

$[-900]_2$ $\{9^2 84^4\}, \{8^4 43^2\}$
$$-q^{10}(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2$$
$$\times(q^{10}+2q^8-4q^7+5q^6-4q^5+5q^4-4q^3+2q^2+1)$$

$[-900]_3$ $\{97^3 4^3\}, \{8^3 5^3 3\}$
$$-q^8(q^2+q+1)^2(q^4+1)(q^4-q^3+2q^2-q+1)(q^4+q^3+q^2+q+1)^2$$
$$\times(q^6+q^4-2q^3+q^2+1)$$

$[-918]$ $\{11\ 7^2 64^2 3\}, \{98^2 65^2 1\}$
$$-q^{10}(q^2+q+1)^3(q^2-q+1)^2(q^2+1)$$
$$\times(2q^{10}+2q^9+2q^8+2q^6+q^5+2q^4+2q^2+2q+2)$$

$[-972]$ $\{11\ 86^2 53^2\}, \{9^2 76^2 41\}$
$$-q^{10}(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4+1)$$
$$\times(2q^6-2q^5+2q^4-q^3+2q^2-2q+2)$$

$[-990]$ $\{10\ 7^2 64^3\}, \{8^3 65^2 2\}$
$$-q^8(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^{14}-q^{13}+3q^{12}-2q^{11}+5q^{10}$$
$$-5q^9+7q^8-5q^7+7q^6-5q^5+5q^4-2q^3-q+1)$$

$[-1053]$ $\{11\ 97543^2\}, \{9^2 87531\}$
$$-q^{10}(q^2+q+1)^4(q^2-q+1)^2$$
$$\times(q^{10}+3q^8-q^7+4q^6-q^5+4q^4-q^3+3q^2+1)$$

$[-1107]$ $\{10\ 976532\}$
$$-q^{10}(q^2+q+1)^3(2q^{16}-4q^{15}+12q^{14}-17q^{13}+33q^{12}-39q^{11}+57q^{10}-57q^9+67q^8-57q^7+57q^6-39q^5+33q^4-17q^3+12q^2-4q+2)$$

$[-1134]$ $\{10\ 985^232\}, \{10\ 97^2432\}$
$$-q^{10}(q^2+q+1)^3(q^2+1)(q^{14}-q^{13}+5q^{12}-6q^{11}+13q^{10}-13q^9+19q^8-15q^7+19q^6-13q^5+13q^4-6q^3+5q^2-q+1)$$

$[-1206]$ $\{11\ 7^265^21\}$
$$-q^{10}(q^2+q+1)^2(q^2+1)(q^{16}+q^{15}+4q^{14}+2q^{13}+6q^{12}+3q^{11}+9q^{10}+3q^9+9q^8+3q^7+9q^6+3q^5+6q^4+2q^3+4q^2+q+1)$$

$[-1215]$ $\{12\ 8654^23\}, \{98^2764\}, \{12\ 76^2542\}, \{10\ 876^25\}$
$$-q^{10}(q^2+q+1)^5(q^2-q+1)^2(q^4-q^3+2q^2-q+1)\times(q^4+q^3+q^2+q+1)$$

$[-1242]$ $\{11\ 7^3532\}, \{10\ 975^31\}$
$$-q^{10}(q^2+q+1)^3(q^2-q+1)(q^{14}+q^{13}+4q^{12}+q^{11}+7q^{10}+q^9+8q^8+8q^6+q^5+7q^4+q^3+4q^2+q+1)$$

$[-1296]$ $\{11\ 8^2543^2\}, \{9^2874^21\}, \{10\ 9863^3\}, \{9^36432\}$
$$-q^{10}(q^2+q+1)^3(q^2+1)^2(q^4+1)(q^8+2q^6-2q^5+4q^4-2q^3+2q^2+1)$$

$[-1350]_1$ $\{11\ 10\ 5^24^23\}, \{98^27^221\}$
$$-q^{10}(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4-q^3+2q^2-q+1)\times(q^4+q^3+q^2+q+1)^2$$

$[-1350]_2$ $\{11\ 6^354^2\}, \{10\ 95^34^2\}$
$$-q^8(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4-q^2+1)(q^4-q^3+2q^2-q+1)\times(q^4+q^3+q^2+q+1)^2$$

$[-1350]_3$ $\{10\ 95^34^2\}, \{8^27^332\}$
$$-q^8(q^2+q+1)^3(q^4+1)(q^4-q^3+2q^2-q+1)^2(q^4+q^3+q^2+q+1)^2$$

$[-1374]$ $\{98^264^23\}$
$$-q^8(q^2+q+1)(q^2+1)(q^{22}+q^{21}+4q^{20}+2q^{19}+11q^{18}+6q^{17}+19q^{16}+6q^{15}+26q^{14}+7q^{13}+29q^{12}+5q^{11}+29q^{10}+7q^9+26q^8+6q^7+19q^6+6q^5+11q^4+2q^3+4q^2+q+1)$$

$[-1377]_1$ $\{11\ 96^24^22\}, \{10\ 8^26^231\}, \{10^2\ 754^22\}, \{10\ 8^2752^2\}$
$$-q^{10}(q^2+q+1)^4(q^2-q+1)^3\times(q^8+2q^7+4q^6+q^5+q^4+q^3+4q^2+2q+1)$$

$[-1377]_2$ $\{10^2\ 65^242\}, \{10\ 87^262^2\}$
$$- q^{10}(q^2+q+1)^3(q^2-q+1)(q^{14}+q^{13}+5q^{12}+q^{11}+7q^{10}+2q^9 + 9q^8-q^7+9q^6+2q^5+7q^4+q^3+5q^2+q+1)$$
$[-1404]$ $\{11\ 86^341\}$
$$- q^{10}(q^2+q+1)^3(q^2+1)(q^4+1) \times (q^{10}+3q^8-q^7+4q^6-q^5+4q^6-q^5+4q^4-q^3+3q^2+1)$$
$[-1422]$ $\{12\ 7^25^23^2\}, \{9^27^25^2\}$
$$- q^{10}(q^2+q+1)^2(q^4+1)(q^{14}+2q^{13}+4q^{12}+4q^{11}+8q^{10}+7q^9 + 10q^8+7q^7+10q^6+7q^5+8q^4+4q^3+4q^2+2q+1)$$
$[-1440]_1$ $\{11\ 10\ 64^33\}, \{98^3621\}$
$$- q^{10}(q^2+q+1)^2(q^2+1)^3(q^4+1)^2(q^4+q^3+q^2+q+1)$$
$[-1440]_2$ $\{11\ 76^2543\}, \{98 76^251\}$
$$- q^8(q^2+q+1)^2(q^2+1)^3(q^4+1)^2(q^4-q^3+2q^2-q+1) \times (q^4+q^3+q^2+q+1)$$
$[-1458]$ $\{11\ 875^242\}, \{10\ 87^2541\}$
$$- q^{10}(q^2+q+1)^5(q^2-q+1)^2(q^2+1) \times (2q^6-2q^5+2q^4-q^3+2q^2-2q+2)$$
$[-1476]$ $\{9^27653^2\}$
$$- q^8(q^2+q+1)^2(q^2+1)(q^4+1)(q^{16}+3q^{14}-q^{13}+8q^{12}-3q^{11} + 11q^{10}-5q^9+13q^8-5q^7+11q^6-3q^5+8q^4-q^3+3q^2+1)$$
$[-1485]$ $\{10\ 8754^3\}, \{8^37542\}$
$$- q^8(q^2+q+1)^3(q^2-q+1)(q^4+q^3+q^2+q+1) \times (q^{14}-q^{13}+3q^{12}-2q^{11}+5q^{10}-5q^9+7q^8-5q^7+7q^6 - 5q^5+5q^4-2q^3-q+1)$$
$[-1548]$ $\{98^25^243\}, \{987^24^23\}$
$$- q^8(q^2+q+1)^2(q^4+1) \times (q^{16}+4q^{14}+7q^{12}-q^{11}+9q^{10}-3q^9+9q^8-3q^7+9q^6 - q^5+7q^4+4q^2+1)$$
$[-1575]_1$ $\{12954^4\}, \{12\ 6^451\}, \{11\ 76^4\}, \{8^473\}$
$$- q^{10}(q^2+q+1)^2(q^4-q^3+2q^2-q+1)(q^4+q^3+q^2+q+1)^2 \times (q^6+q^5+q^4+q^3+q^2+q+1)$$
$[-1575]_2$ $\{10^3\ 3^4\}, \{9^42^3\}$
$$- q^{12}(q^2+q+1)^2(q^4+q^3+q^2+q+1)^2(q^6+q^4-2q^3+q^2+1)$$

$[-1620]$ $\{9^2754^3\}, \{8^3753^2\}$
$$-q^8(q^2+q+1)^4(q^2-q+1)^2(q^4+1)$$
$$\times(q^4+q^3+q^2+q+1)(q^6+q^4-2q^3+q^2+1)$$

$[-1647]$ $\{10^2\ 6^243^2\}, \{9^286^22^2\}$
$$-q^{10}(q^2+q+1)^3(q^2-q+1)^2(q^{12}+3q^{11}+7q^{10}+6q^9+6q^8+4q^7$$
$$+7q^6+4q^5+6q^4+6q^3+7q^2+3q+1)$$

$[-1656]$ $\{10\ 7^26^23^2\}, \{9^26^25^22\}$
$$-q^8(q^2+q+1)^2(q^4+1)^2(q^{14}+q^{13}+4q^{12}+q^{11}+7q^{10}+q^9$$
$$+8q^8+8q^6+q^5+7q^4+q^3+4q^2+q+1)$$

$[-1674]$ $\{10\ 876542\}$
$$-q^8(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^{16}-q^{15}+4q^{14}-3q^{13}$$
$$+9q^{12}-6q^{11}+13q^{10}-9q^9+15q^8-9q^7+13q^6-6q^5+9q^4$$
$$-3q^3+4q^2-q+1)$$

$[-1728]$ $\{11\ 865^243\}, \{987^2641\}$
$$-q^8(q^2+q+1)^3(q^2-q+1)(q^2+1)^3(q^4+1)^3$$

$[-1800]_1$ $\{11\ 85^43\}, \{97^441\}$
$$-q^8(q^2+q+1)^2(q^2+1)(q^4+1)^2(q^4-q^3+2q^2-q+1)$$
$$\times(q^4+q^3+q^2+q+1)^2$$

$[-1800]_2$ $\{97^2654^2\}, \{8^2765^23\}$
$$-q^6(q^2+q+1)^2(q^2+1)(q^4+1)^2(q^4-q^3+2q^2-q+1)^2$$
$$\times(q^4+q^3+q^2+q+1)^2$$

$[-1800]_3$ $\{8^364^3\}$
$$-q^8(q^2+q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)^2$$
$$\times(q^{10}+2q^8-4q^7+5q^6-4q^5+5q^4-4q^3+2q^2+1)$$

$[-1944]$ $\{10\ 965^243\}, \{987^2632\}$
$$-q^8(q^2+q+1)^4(q^2-q+1)(q^2+1)^2(q^4+1)$$
$$\times(q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$$

$[-1998]$ $\{10^2\ 84^23^2\}, \{9^28^242^2\}$
$$-q^{10}(q^2+q+1)^3(q^2-q+1)(q^4+1)$$
$$\times(q^{10}+2q^9+4q^8+3q^7+6q^6+5q^5+6q^4+3q^3+4q^2+2q+1)$$

$[-2133]$ $\{10\ 9^253^3\}, \{9^373^22\}$
$$-q^{10}(q^2+q+1)^3(q^2-q+1)(q^{14}+2q^{13}+4q^{12}+4q^{11}$$
$$+8q^{10}+7q^9+10q^6+7q^5+8q^4+4q^3+4q^2+2q+1)$$

$[-2160]_1$ $\{11\ 8654^3\}, \{8^37641\}$
$-q^8(q^2+q+1)^3(q^2+1)^2(q^4+1)^2(q^4-q^3+2q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[-2160]_2$ $\{986^25^23\}, \{97^26^243\}$
$-q^6(q^2+q+1)^3(q^2-q+1)(q^2+1)(q^4+1)^3(q^4-q^3+2q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[-2268]$ $\{10\ 87653^2\}, \{9^276542\}$
$-q^8(q^2+q+1)^4(q^2-q+1)(q^2+1)(q^4+1)$
$\times (q^{10}-q^9+4q^8-4q^7+6q^6-5q^5+6q^4-4q^3+4q^2-q+1)$

$[-2421]$ $\{9^26^33^2\}$
$-q^8(q^2+q+1)^2(q^{22}+2q^{21}+5q^{20}+4q^{19}+10q^{18}$
$+8q^{17}+18q^{16}+10q^{15}+26q^{14}+30q^{12}+13q^{11}$
$+30q^{10}+14q^9+26q^8+10q^7+18q^6+8q^5+10q^4+4q^3+5q^2+2q+1)$

$[-2430]$ $\{11\ 76^25^22\}, \{10\ 7^26^251\}$
$-q^8(q^2+q+1)^5(q^2-q+1)^2(q^4+1)(q^4-q^3+q^2-q+1)$
$\times (q^4+q^3+q^2+q+1)$

$[-2484]$ $\{10\ 875^32\}, \{10\ 7^3542\}$
$-q^8(q^2+q+1)^3(q^2-q+1)(q^4+1)(q^{14}+q^{13}+4q^{12}+q^{11}$
$+7q^{10}+q^9+8q^8+8q^6+q^5+7q^4+q^3+4q^2+q+1)$

$[-2754]$ $\{10\ 86^342\}$
$-q^8(q^2+q+1)^3(q^2-q+1)(q^4+1)(q^{14}+q^{13}+5q^{12}+q^{11}$
$+7q^{10}+2q^9+9q^8-q^7+9q^6+2q^5+7q^4+q^3+5q^2+q+1)$

$[-2835]_1$ $\{11^2\ 4^5\}, \{8^51^2\}$
$-q^{10}(q^2+q+1)^3(q^4+q^3+q^2+q+1)(q^6+q^3+1)$
$\times (q^6+q^5+q^4+q^3+q^2+q+1)$

$[-2835]_2$ $\{87^25^4\}, \{7^45^24\}$
$-q^6(q^2+q+1)^4(q^2-q+1)(q^4-q^2+1)(q^4+q^3+q^2+q+1)$
$\times (q^6-q^5+q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$

$[-2907]$ $\{10\ 96^24^23\}, \{98^26^232\}$
$-q^8(q^2+q+1)^2(q^{22}+q^{21}+5q^{20}+3q^{19}+14q^{18}+5q^{17}$
$+27q^{16}+7q^{15}+40q^{14}+7q^{13}+48q^{12}+7q^{11}+48q^{10}+7q^9$
$+40q^8+7q^7+27q^6+5q^5+14q^4+3q^3+5q^2+q+1)$

$[-2916]$ $\{11\ 7^25^243\}, \{9 8 7^2 5^2 1\}$

$$- q^8(q^2+q+1)^5(q^2-q+1)(q^4+1)$$
$$\times (q^8-q^7+2q^6-q^5+q^4-q^3+2q^2-q+1)$$

$[-3150]$ $\{12\ 75^34^2\}, \{12\ 6^25^33\}, \{97^36^2\}, \{8^27^35\}$

$$- q^8(q^2+q+1)^2(q^4+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$$
$$\times (q^6+q^5+q^4+q^3+q^2+q+1)$$

$[-3240]_1$ $\{116^452\}, \{1076^41\}$

$$- q^8(q^2+q+1)^4(q^2-q+1)(q^2+1)^2(q^4+1)(q^4-q^2+1)$$
$$\times (q^4+q^3+q^2+q+1)$$

$[-3240]_2$ $\{9865^34\}, \{87^3643\}$

$$- q^6(q^2+q+1)^4(q^2-q+1)(q^2+1)^2(q^4+1)(q^4-q^2+1)$$
$$\times (q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)$$

$[-3564]$ $\{10\ 8^25^23^2\}, \{9^27^24^22\}$

$$- q^8(q^2+q+1)^4(q^2-q+1)^2(q^4+1)$$
$$\times (q^{10}+2q^9+3q^8+4q^6+2q^5+4q^4+3q^2+2q+1)$$

$[-3690]$ $\{97^25^34\}, \{87^35^23\}$

$$- q^6(q^2+q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)$$
$$\times (q^{18}+3q^{16}+5q^{14}+6q^{12}-q^{11}+7q^{10}-q^9+7q^8-q^7+6q^6$$
$$+5q^4+3q^2+1)$$

$[-3726]$ $\{10\ 9754^23\}, \{98^27532\}$

$$- q^8(q^2+q+1)^4(q^2-q+1)^2(q^{14}+q^{13}+4q^{12}+q^{11}+7q^{10}+q^9$$
$$+8q^8+8q^6+q^5+7q^4+q^3+4q^2+q+1)$$

$[-4050]_1$ $\{11\ 95^24^3\}, \{8^37^231\}$

$$- q^8(q^2+q+1)^4(q^2-q+1)(q^4+1)(q^4-q^3+q^2-q+1)$$
$$\times (q^4+q^3+q^2+q+1)^2$$

$[-4050]_2$ $\{10\ 76^254^2\}, \{8^276^252\}$

$$- q^6(q^2+q+1)^4(q^2-q+1)(q^4+1)(q^4-q^3+q^2-q+1)^2$$
$$\times (q^4+q^3+q^2+q+1)^2$$

$[-4050]_3$ $\{8^27654^2\}$

$$- q^6(q^2+q+1)^3(q^2+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$$
$$\times (q^{10}-q^9+2q^8-2q^7+2q^6-q^5+2q^4-2q^3+2q^2-q+1)$$

$[-4410]$ $\{10\ 75^5\}, \{7^552\}$

$$- q^6(q^2+q+1)^2(q^4+1)(q^4+q^3+q^2+q+1)$$
$$\times (q^6-q^5+q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)^2$$

$[-4566]$ $\{9^28643^2\}$

$$- q^8(q^2+q+1)(q^{24}+3q^{23}+9q^{22}+14q^{21}+30q^{20}+37q^{19}$$
$$+ 68q^{18}+68q^{17}+109q^{16}+95q^{15}+142q^{14}+109q^{13}+152q^{12}$$
$$+ 109q^{11}+142q^{10}+95q^9+109q^8+68q^7+68q^6+37q^5+30q^4$$
$$+ 14q^3+9q^2+3q+1)$$

$[-5310]$ $\{986^343\}$

$$- q^6(q^2+q+1)^2(q^2+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)$$
$$\times (q^{16}+q^{15}+3q^{14}+2q^{13}+5q^{12}+3q^{11}+8q^{10}+3q^9$$
$$+ 7q^8+3q^7+8q^6+3q^5+5q^4+2q^3+3q^2+q+1)$$

$[-6075]_1$ $\{10^2\ 64^4\}, \{8^462^2\}$

$$- q^8(q^2+q+1)^4(q^2-q+1)(q^4+q^3+q^2+q+1)^2$$
$$\times (q^8-q^5+3q^4-q^3+1)$$

$[-6075]_2$ $\{10\ 6^453\}, \{976^42\}$

$$- q^6(q^2+q+1)^5(q^2-q+1)^2(q^4-q^2+1)(q^4-q^3+q^2-q+1)$$
$$\times (q^4+q^3+q^2+q+1)^2$$

$[-6210]$ $\{10\ 76^25^23\}, \{97^26^252\}$

$$- q^6(q^2+q+1)^3(q^2-q+1)(q^4+1)(q^4+q^3+q^2+q+1)$$
$$(q^{14}+3q^{12}+4q^{10}-q^9+5q^8-q^7+5q^6-q^5+4q^4+3q^2+1)$$

$[-7110]$ $\{10\ 7^25^33\}, \{97^35^22\}$

$$- q^6(q^2+q+1)^2(q^4+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)$$
$$(q^{14}+2q^{13}+4q^{12}+4q^{11}+8q^{10}+7q^9+10q^8+7q^7+10q^6$$
$$+ 7q^5+8q^4+4q^3+4q^2+2q+1)$$

$[-7560]$ $\{11\ 6^25^34\}, \{87^36^21\}$

$$- q^6(q^2+q+1)^3(q^2-q+1)(q^2+1)^2(q^4+1)(q^4-q^2+1)$$
$$\times (q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$$

$[-7875]$ $\{11\ 75^44\}, \{87^451\}$

$$- q^6(q^2+q+1)^2(q^4-q^3+q^2-q+1)^2(q^4+q^3+q^2+q+1)^3$$
$$\times (q^6+q^5+q^4+q^3+q^2+q+1)$$

$[-8064]$ $\{9876543\}$

$$- q^6(q^2+q+1)^2(q^2+1)^4(q^4+1)^2$$
$$\times (q^{10}+q^9+3q^8+2q^6+2q^4+3q^2+q+1)$$

[−8505] $\{10\ 865^24^2\}, \{8^27^2642\}$
$$- q^6(q^2+q+1)^5(q^2-q+1)^2(q^4+q^3+q^2+q+1)$$
$$\times (q^{12}+q^{10}-2q^9+4q^8-q^7+q^6-q^5+4q^4-2q^3+q^2+1)$$

[−10395] $\{12\ 5^6\}, \{7^6\}$
$$- q^6(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^6+q^3+1)$$
$$\times (q^6+q^5+q^4+q^3+q^2+q+1)(q^{10}+q^9+q^8+q^7+q^6+q^5+q^4$$
$$+ q^3+q^2+q+1)$$

[−11025] $\{876^25^3\}, \{7^36^254\}$
$$- q^4(q^2+q+1)^2(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^2$$
$$\times (q^6-q^5+q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)^2$$

[−11115] $\{9^26^24^3\}, \{8^36^23^2\}$
$$- q^6(q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^{22}+2q^{21}+4q^{20}+4q^{19}$$
$$+ 9q^{18}+8q^{17}+16q^{16}+9q^{15}+24q^{14}+13q^{13}+28q^{12}+11q^{11}$$
$$+ 28q^{10}+13q^9+24q^8+9q^7+16q^6+8q^5+9q^4+4q^3+4q^2+2q+1)$$

[−12600] $\{96^454\}, \{876^43\}$
$$- q^4(q^2+q+1)^2(q^2+1)^2(q^4-q^2+1)(q^4-q^3+q^2-q+1)$$
$$\times (q^4+q^3+q^2+q+1)^2(q^6+q^5+q^4+q^3+q^2+q+1)$$

[−18225] $\{976^25^24\}, \{87^26^253\}$
$$- q^4(q^2+q+1)^6(q^2-q+1)^3(q^4-q^2+1)(q^4-q^3+q^2-q+1)$$
$$\times (q^4+q^3+q^2+q+1)^2$$

[−20250] $\{8^26^34^2\}$
$$- q^4(q^2+q+1)^3(q^4+1)(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)^3$$
$$\times (q^8-q^5+3q^4-q^3+1)$$

[−23625] $\{10\ 6^25^4\}, \{7^46^22\}$
$$- q^4(q^2+q+1)^3(q^2-q+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1)$$
$$\times (q^4+q^3+q^2+q+1)^3$$

[−42525] $\{86^45^2\}, \{7^26^44\}$
$$- q^2(q^2+q+1)^4(q^2-q+1)(q^4-q^2+1)(q^4-q^3+q^2-q+1)$$
$$\times (q^4+q^3+q^2+q+1)^2(q^6+q^3+1)(q^6+q^5+q^4+q^3+q^2+q+1)$$

[−135135] $\{6^7\}$
$$- (q^2+q+1)^2(q^4+q^3+q^2+q+1)(q^6+q^3+1)$$
$$\times (q^6+q^5+q^4+q^3+q^2+q+1)$$
$$\times (q^{10}+q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1)$$
$$\times (q^{12}+q^{11}+q^{10}+q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1)$$

10. Sum of squares of the coefficients for the second power of the Vandermonde

Di Francesco etal give a remarkable formula for the sum of the squares of the coefficients of the powers of the Vandermonde determinant. For the second power they give (5.14)

$$\frac{(3N)!}{N!(3!)^N} \tag{10-1}$$

For $N = 2..6$ we have for the $q-$polynomials:-

$$N = 2\ q^4 + 2q^3 + 4q^2 + 2q + 1$$

$$\begin{aligned} N = 3\ & q^{12} + 4q^{11} + 11q^{10} + 20q^9 + 34q^8 + 44q^7 + 52q^6 + 44q^5 + 34q^4 + 20q^3 \\ & + 11q^2 + 4q + 1 \end{aligned}$$

$$\begin{aligned} N = 4\ & q^{24} + 6q^{23} + 22q^{22} + 58q^{21} + 128q^{20} + 242q^{19} + 418q^{18} + 646q^{17} \\ & + 929q^{16} + 1210q^{15} + 1490q^{14} + 1670q^{13} + 1760q^{12} + 1670q^{11} + 1490q^{10} \\ & + 1210q^9 + +929q^8 + 646q^7 + 418q^6 + 242q^5 + 128q^4 + 58q^3 + 22q^2 + 6q + 1 \end{aligned}$$

$$\begin{aligned} N = 5\ & q^{40} + 8q^{39} + 37q^{38} + 124q^{37} + 339q^{36} + 796q^{35} + 1671q^{34} + 3192q^{33} \\ & + 5662q^{32} + 9392q^{31} + 14755q^{30} + 21946q^{29} + 31190q^{28} + 42202q^{27} \\ & + 54902q^{26} + 68238q^{25} + 81835q^{24} + 93846q^{23} + 104006q^{22} \\ & + 110180q^{21} + 112756q^{20} + 110180q^{19} + 104006q^{18} + 93846q^{17} \\ & + 81835q^{16} + 68238q^{15} + 54902q^{14} + 42202q^{13} + 31190q^{12} \\ & + 21946q^{11} + 14773q^{10} + 9392q^9 + 5662q^8 + 3192q^7 + 1671q^6 + 796q^5 \\ & + 339q^4 + 124q^3 + 37q^2 + 8q + 1 \end{aligned}$$

$$\begin{aligned} N = 6\ & q^{60} + 10q^{59} + 56q^{58} + 226q^{57} + 735q^{56} + 2040q^{55} + 5016q^{54} + 11184q^{53} \\ & + 23014q^{52} + 44236q^{51} + 80248q^{50} + 138356q^{49} + 228178q^{48} + 361328q^{47} \\ & + 551776q^{46} + 814112q^{45} + 1164421q^{44} + 1615466q^{43} + 2180352q^{42} \\ & + 2861762q^{41} + 3663515q^{40} + 4568012q^{39} + 5565064q^{38} + 6609308q^{37} \\ & + 7679162q^{36} + 8701940q^{35} + 9654832q^{34} + 10448380q^{33} + 11075390q^{32} \\ & + 11449688q^{31} + 11594784q^{30} + 11449688q^{29} + 11075390q^{28} + 10448389q^{27} \\ & + 9654832q^{26} + 8701940q^{25} + 7679162q^{24} + 6609308q^{23} + 5565064q^{22} \\ & + 4568012q^{21} + 3663515q^{20} + 2861762q^{19} + 2180352q^{18} + 1615466q^{17} \\ & + 1164421q^{16} + 814112q^{15} + 551776q^{14} + 361328q^{13} + 228178q^{12} \\ & + 138356q^{11} + 80248q^{10} + 44236q^9 + 23014q^8 + 11184q^7 + 5016q^6 + 2040q^5 \\ & + 735q^4 + 226q^3 + 56q^2 + 10q + 1 \end{aligned}$$

We note that in each case the coefficient distribution is symmetric and unimodal. This suggests that one should be able to obtain a q-analogue of (10-1) perhaps in terms of q-binomials?

11. $q-$expansion for some $N = 9$ cases

$$\begin{aligned}\{16\ 13\ 11\ 985^2 41\}\ Q1 = {} & q^{50} - 13q^{49} + 82q^{48} - 338q^{47} + 1035q^{46} - 2535q^{45} \\ & + 5210q^{44} - 9282q^{43} + 14659q^{42} - 20839q^{41} + 26940q^{40} \\ & - 31876q^{39} + 34646q^{38} - 34646q^{37} + 31876q^{36} - 26940q^{35} \\ & + 20839q^{34} - 14659q^{33} + 9282q^{32} - 5210q^{31} + 2535q^{30} \\ & - 1035q^{29} + 338q^{28} - 82q^{27} + 13q^{26} - q^{25}\end{aligned}$$

$$\begin{aligned}\{16\ 13\ 11\ 9854^2 2\}\ Q2 = {} & -q^{55} + 10q^{54} - 48q^{53} + 153q^{52} - 381q^{51} + 816q^{50} \\ & - 1570q^{49} + 2749q^{48} - 4416q^{47} + 6584q^{46} - 9200q^{45} + 12096q^{44} \\ & - 14986q^{43} + 17548q^{42} - 19488q^{41} + 20546q^{40} - 20546q^{39} \\ & + 19488q^{38} - 17548q^{37} + 14986q^{36} - 12096q^{35} + 9200q^{34} \\ & - 6584q^{33} + 4416q^{32} - 2749q^{31} + 1570q^{30} - 816q^{29} + 381q^{28} \\ & - 153q^{27} + 48q^{26} - 10q^{25} + q^{24}\end{aligned}$$

$$\begin{aligned}\{16\ 13\ 11\ 976541\}\ Q3 = {} & -q^{57} + 10q^{56} - 50q^{55} + 173q^{54} - 477q^{53} + 1122q^{52} \\ & - 2331q^{51} + 4371q^{50} - 7509q^{49} + 11939q^{48} - 17699q^{47} \\ & + 24601q^{46} - 32196q^{45} + 39797q^{44} - 46567q^{43} + 51664q^{42} \\ & - 54405q^{41} + 54405q^{40} - 51664q^{39} + 46567q^{38} - 39797q^{37} \\ & + 32196q^{36} - 24601q^{35} + 17699q^{34} - 11939q^{33} + 7509q^{32} \\ & - 4371q^{31} + 2331q^{30} - 1122q^{29} + 477q^{28} - 173q^{27} + 50q^{26} \\ & - 10q^{25} + q^{24}\end{aligned}$$

$$\begin{aligned}\{16\ 12\ 11\ 97^2 4^2 2\}\ Q4 = {} & -q^{59} + 9q^{58} - 40q^{57} + 126q^{56} - 335q^{55} + 798q^{54} \\ & - 1708q^{53} + 3307q^{52} - 5900q^{51} + 9834 * q^{50} - 15379q^{49} \\ & + 22618q^{48} - 31444q^{47} + 41534q^{46} - 52230q^{45} + 62563q^{44} \\ & - 71507q^{43} + 78158q^{42} - 81741q^{41} + 81741q^{40} - 78158q^{39} \\ & + 71507q^{38} - 62563q^{37} + 52230q^{36} + 41534q^{35} + 31444q^{34} \\ & - 22618q^{33} + 15379q^{32} - 9834q^{31} + 5900q^{30} - 3307q^{29} \\ & + 1708q^{28} - 798q^{27} + 335q^{26} - 126q^{25} + 40q^{24} - 9q^{23} + q^{22}\end{aligned}$$

12. The reversal pairs of $N = 9$ admissible partitions having null coefficients

$\{16\ 13\ 11\ 985^{2}41\}$	$\{15\ 12\ 11^{2}\ 8753\}$
$\{16\ 13\ 11\ 9854^{2}2\}$	$\{14\ 12^{2}\ 11\ 8753\}$
$\{16\ 13\ 11\ 976541\}$	$\{15\ 12\ 11\ 10\ 9753\}$
$\{16\ 13\ 10\ 9^{2}6531\}$	$\{15\ 13\ 11\ 10\ 7^{2}63\}$
$\{16\ 13\ 10\ 987531\}$	$\{15\ 13\ 11\ 98763\}$
$\{16\ 12\ 11\ 97^{2}4^{2}2\}$	$\{14\ 12^{2}\ 9^{2}754\}$
$\{16\ 12\ 10^{2}\ 96531\}$	$\{15\ 13\ 11\ 10\ 76^{2}4\}$
$\{16\ 12\ 10^{2}\ 7^{2}532\}$	$\{14\ 13\ 11\ 9^{2}6^{2}4\}$
$\{15\ 14\ 11\ 985^{2}41\}$	$\{15\ 12\ 11^{2}\ 87521\}$
$\{15\ 14\ 11\ 9854^{2}2\}$	$\{14\ 12^{2}\ 11\ 87521\}$
$\{15\ 14\ 11\ 976541\}$	$\{15\ 12\ 11\ 10\ 97521\}$
$\{15\ 14\ 10\ 9^{2}6531\}$	$\{15\ 13\ 11\ 10\ 7^{2}621\}$
$\{15\ 14\ 10\ 987531\}$	$\{15\ 13\ 11\ 987621\}$
$\{15\ 13\ 11\ 10\ 7^{2}52^{2}\}$	$\{14^{2}11\ 9^{2}6531\}$
$\{15\ 13\ 11\ 9^{2}652^{2}\}$	$\{14^{2}11\ 10\ 7^{2}531\}$
$\{15\ 13\ 11\ 9^{2}64^{2}1\}$	$\{15\ 12^{2}\ 10\ 7^{2}531\}$
$\{15\ 13\ 11\ 9^{2}6432\}$	$\{14\ 13\ 12\ 10\ 7^{2}531\}$
$\{15\ 13\ 11\ 976^{2}32\}$	$\{14\ 13\ 10^{2}\ 97531\}$
$\{15\ 13\ 11\ 976542\}$	$\{14\ 12\ 11\ 10\ 97531\}$
$\{15\ 13\ 10^{2}\ 85^{2}42\}$	$\{14\ 12\ 11^{2}\ 86^{2}31\}$
$\{15\ 13\ 9^{2}7^{2}543\}$	$\{13\ 12\ 11\ 9^{2}7^{2}31\}$
$\{15\ 12^{2}\ 9^{2}5^{2}41\}$	$\{15\ 12\ 11^{2}\ 7^{2}4^{2}1\}$
$\{15\ 12^{2}\ 9^{2}54^{2}2\}$	$\{14\ 12^{2}\ 11\ 7^{2}4^{2}1\}$
$\{15\ 12\ 11^{2}\ 7^{2}432\}$	$\{14\ 13\ 12\ 9^{2}5^{2}41\}$
$\{15\ 12\ 11\ 10\ 76^{2}41\}$	$\{15\ 12\ 10^{2}\ 96541\}$
$\{15\ 10\ 9^{3}5^{4}\}$	$\{11^{4}7^{3}61\}$
$\{14\ 13\ 12\ 9^{2}54^{2}2\}$	$\{14\ 12^{2}\ 11\ 7^{2}432\}$
$\{13^{3}\ 10\ 7643^{2}\}$	$\{13^{2}\ 12\ 10\ 963^{3}\}$
$\{13\ 12\ 10^{2}\ 95^{3}3\}$	$\{13\ 11^{3}\ 76^{2}43\}$
$\{12^{2}\ 987^{3}64\}$	$\{12\ 10\ 9^{3}874^{2}\}$
$\{12\ 10^{3}\ 76^{3}5\}$	$\{11\ 10^{3}\ 96^{3}4\}$
$\{12\ 10\ 9^{3}85^{3}\}$	$\{11^{3}\ 87^{3}64\}$
$\{11^{3}\ 976^{3}5\}$	$\{11\ 10^{3}\ 975^{3}\}$

13. Absolute Sum of the coefficients $c_\lambda(q)$

We define

$$QN_A = \sum_\lambda |c_\lambda(q) \qquad (13\text{-}1)$$

and give the polynomials for $N = 2, \ldots, 5$ below

$$Q2_A = q^2 + 2q + 1$$

$$Q3_A = q^6 + 3q^5 + 6q^4 + 8q^3 + 6q^2 + 3q + 1$$

$$\begin{aligned} Q4_A = {} & q^{12} + 4q^{11} + 10q^{10} + 22q^9 + 35q^8 + 48q^7 + 52q^6 + 48q^5 + 35q^4 \\ & + 22q^3 + 10q^2 + 4q + 1 \end{aligned}$$

$$\begin{aligned} Q5_A = {} & q^{20} + 5q^{19} + 15q^{18} + 37q^{17} + 78q^{16} + 145q^{15} + 230q^{14} + 336q^{13} + 429q^{12} \\ & + 510q^{11} + 530q^{10} + 510q^9 + 429q^8 + 336q^7 + 230q^6 + 145q^5 + 78q^4 \\ & + 37q^3 + 15q^2 + 5q + 1 \end{aligned}$$

$$\begin{aligned} Q6_A = {} & q^{30} + 6q^{29} + 21q^{28} + 58q^{27} + 136q^{26} + 290q^{25} + 560q^{24} \\ & + 994q^{23} + 1621q^{22} + 2450q^{21} + 3455q^{20} + 4558q^{19} + 5666q^{18} \\ & + 6598q^{17} + 7246q^{15} + 7246q^{14} + 6598q^{13} + 5666q^{12} + 4558q^{11} \\ & + 3455q^{10} + 2450q^9 + 1621q^8 + 994q^7 + 560q^6 + 290q^5 + 136q^4 + 58q^3 \\ & + 21q^2 + 6q + 1 \end{aligned}$$

14. Sums of coefficients $c_\lambda(q)$ for $\Delta^4(q)$

$$Q^2S(N) = \sum_\mu c_\mu(q) \qquad (14\text{-}1)$$

where

$$\begin{aligned} \Delta^4(q) &= \prod_{1\le i,\ne j}^{N} ((x_i - qx_j)(qx_i - x_j))^2 \\ &= \sum_\mu c_\mu(q) s_\mu(1) \qquad (14\text{-}2) \end{aligned}$$

$$Q^2S(2) = q^4 + 4q^2 + 1$$

$$Q^2S(3) = q^{12} + 2q^{11} + 14q^{10} + 10q^9 + 55q^8 + 18q^7 + 86q^6 + 18q^5 + 55q^4 + 10q^3 + 14q^2 + 2q + 1$$

$$Q^2S(4) = q^{24} + 4q^{23} + 22q^{22} + 28q^{21} + 142q^{20} + 20q^{19} + 608q^{18} - 268q^{17} + 1827q^{16} - 1144q^{15} + 3634q^{14} - 2048q^{13} + 4644q^{12} - 2048q^{11} + 3634q^{10} - 1144q^9 + 1827q^8 - 268q^7 + 608q^6 + 20q^5 + 142q^4 + 28q^3 + 22q^2 + 4q + 1$$

Explicit calculation gives for $q = 1$

$$N = 2 \quad 6 = (2)(3)$$

$$N = 3 \quad 286 = (2)(11)(13)$$

$$N = 4 \quad 10296 = (2)^3(3)^2(11)(13)$$

$$N = 5 \quad 1657656 = (2)^3(3)^2(7)(11)(13)(23)$$

$$N = 6 \quad 150846696 = (2)^3(3)^2(7)^2(11)(13)^2(23)$$

15. Powers of the Vandermonde in two variables

If $\Delta(x_1, x_2) = x_1 - x_2$ then in terms of $S-$functions

$$\Delta^2 = \{2\} - 3\{1^2\} \tag{15-1}$$

For Δ^{2k} we have

$$\Delta^{2k} = (\{2\} - 3\{1^2\})^k \tag{15-2}$$

The sum of the coefficients, $S(k)$ for $k = 1, 2, \ldots$ leads to the integer sequence

$$-2, 6, -20, 70, -252, \ldots \tag{15-3}$$

and from Sloane's On-Line Encyclopedia of integer sequences
http://www.research.att.com/∼njas/sequences/

we find that (Sloane's A000984)

$$S(k) = (-1)^k \frac{(2k)!}{(k!)^2} \tag{15-4}$$

The sum of the absolute coefficients, $A(k) = \sum_\lambda |c_\lambda|$, appears to be

$$A(k) = 2^{2k} \tag{15-5}$$

I have not been able to establish this result. One might note that in terms of SO_3 we have

$$\Delta^2 = [1] - 3[0]$$

and hence

$$\Delta^{2k} = \sum_{n=0}^{k} [1]^{k-n}(-3)^n \binom{k}{n} \tag{15-6}$$

The number of SO_3 irreps contained in $[1]^N$ is the largest coefficient in

$$(1 + x + x^2)^N = 1, 1, 3, 7, 19, 51, \ldots \tag{15-7}$$

which is Sloane's A002426 series.

Derivations of (15-4) and (15-5) have recently been given by R C King (private communication September 2002).

16. $q = -1$ Expansions

The appearance of $q-$polynomials with negative coefficients in the Vandermonde expansion seemed, initially, surprising but leads to the vanishing of some terms when $q = 1$. In the case of $q = -1$ the unimodal symmetric polynomials all equate to unity.

Consider

$$\Delta_N = (-1)^N \prod_{i<j}^{N} (x_i + x_j) \tag{16-1}$$

This is just the monomial content of the staircase Schur-function

$$\Delta_N = (-1)^N \{N-1, N-2, \ldots, 1\} \tag{16-2}$$

Hence

$$\Delta_N^p = (-1)^{Np} (\{N-1, N-2, \ldots, 1\})_N^p \tag{16-3}$$

The number of admissible tableaux appearing in (16-3) is precisely the number of distinct tableaux that appear in the $U(N)$ Kronecker $p-$th power of the appropriate staircase partition.

Thus for $N = 8$ explicit calculation of the Kronecker square of $\{7654321\}$ yields 5302 distinct tableaux in agreement with Di Francesco etal. In particular we find for the 8 partitions having vanishing coefficients for $q = 1$ the multiplicities given right most below:-

$$\{13\ 11985^241\}, \quad \{13\ 10\ 9^26531\} \quad (576) \tag{16-4a}$$
$$\{13\ 11\ 9854^22\}, \quad \{12\ 10^2\ 96531\} \quad (864) \tag{16-4b}$$
$$\{13\ 11\ 976541\}, \quad \{13\ 10\ 987531\} \quad (1152) \tag{16-4c}$$
$$\{12\ 11\ 97^24^22\}, \quad \{12\ 10^2\ 7^2532\} \quad (1872) \tag{16-4d}$$

Those multiplicities follow directly from putting $q = -1$ in the corresponding $q-$polynomials given below:-

$$-q^{17}(q^2-q+1)^2(q^2+1)^2(q-1)^4(q^2+q+1)^5 \tag{16-4a$'$}$$
$$+q^{16}(q^2+1)(q^2-q+1)^3(q-1)^4(q^2+q+1)^6 \tag{16-4b$'$}$$
$$+q^{16}(q^2-q+1)^2(q^2+1)^3(q-1)^4(q^2+q+1)^5 \tag{16-4c$'$}$$
$$+q^{14}(q^{10}+q^9+3q^8+4q^6+q^5+4q^4+3q^2+q+1)(q^2-q+1)^2$$
$$(q-1)^4(q^2+q+1)^5 \tag{16-4d$'$}$$

For convenience we now tabulate the Kronecker squares for $N = 1, \ldots, 5$

N	Kronecker	Square			
2	$\{2\}$	$+\{1^2\}$			
3	$\{42\}$	$+\{41^2\}$	$+\{3^2\}$	$+2\{321\}$	$+\{2^3\}$
4	$\{642\}$	$+\{641^2\}$	$+\{63^2\}$	$+2\{6321\}$	$+\{62^3\}$
	$+\{5^22\}$	$+\{5^21^2\}$	$+2\{543\}$	$+4\{5421\}$	$+3\{53^21\}$
	$+3\{532^2\}$	$+\{4^3\}$	$+3\{4^231\}$	$+2\{4^22^2\}$	$+3\{43^22\}$
	$+\{3^4\}$				
5	$\{8642\}$	$+\{8641^2\}$	$+\{863^2\}$	$+2\{86321\}$	$+\{862^3\}$
	$+\{85^22\}$	$+\{85^21^2\}$	$+2\{8543\}$	$+4\{85421\}$	$+3\{853^21\}$
	$+3\{8532^2\}$	$+\{84^3\}$	$+3\{84^231\}$	$+2\{84^22^2\}$	$+3\{843^22\}$
	$+\{83^4\}$	$+\{7^242\}$	$+\{7^241^2\}$	$+\{7^23^2\}$	$+2\{7^2321\}$
	$+\{7^22^3\}$	$+2\{7652\}$	$+2\{7651^2\}$	$+4\{7643\}$	$+8\{76421\}$
	$+6\{763^21\}$	$+6\{7632^2\}$	$+3\{75^23\}$	$+6\{75^221\}$	$+3\{754^2\}$
	$+12\{75431\}$	$+9\{7542^2\}$	$+9\{753^22\}$	$+4\{74^31\}$	$+8\{74^232\}$
	$+4\{743^3\}$	$+\{6^32\}$	$+\{6^31^2\}$	$+3\{6^253\}$	$+6\{6^2521\}$
	$+2\{6^24^2\}$	$+9\{6^2431\}$	$+7\{6^242^2\}$	$+6\{6^23^22\}$	$+3\{65^24\}$
	$+9\{65^231\}$	$+6\{65^22^2\}$	$+8\{654^21\}$	$+16\{65432\}$	$+5\{653^3\}$
	$+6\{64^32\}$	$+6\{64^23^2\}$	$+\{5^4\}$	$+4\{5^341\}$	$+5\{5^332\}$
	$+6\{5^24^22\}$	$+5\{5^243^2\}$	$+4\{54^33\}$	$+\{4^5\}$	

and for $N = 6$

$\{10\ 8642\}$	$+\ \{10\ 8641^2\}$	$+\ \{10\ 863^2\}$	$+\ 2\{10\ 86321\}$	$+\ \{10\ 862^3\}$
$+\ \{10\ 85^22\}$	$+\ \{10\ 85^21^2\}$	$+\ 2\{10\ 8543\}$	$+\ 4\{10\ 85421\}$	$+\ 3\{10\ 853^21\}$
$+\ 3\{10\ 8532^2\}$	$+\ \{10\ 84^3\}$	$+\ 3\{10\ 84^231\}$	$+\ 2\{10\ 84^22^2\}$	$+\ 3\{10\ 843^22\}$
$+\ \{10\ 83^4\}$	$+\ \{10\ 7^242\}$	$+\ \{10\ 7^241^2\}$	$+\ \{10\ 7^23^2\}$	$+\ 2\{10\ 7^2321\}$
$+\ \{10\ 7^22^3\}$	$+\ 2\{10\ 7652\}$	$+\ 2\{10\ 7651^2\}$	$+\ 4\{10\ 7643\}$	$+\ 8\{10\ 76421\}$
$+\ 6\{10\ 763^21\}$	$+\ 6\{10\ 7632^2\}$	$+\ 3\{10\ 75^23\}$	$+\ 6\{10\ 75^221\}$	$+\ 3\{10\ 754^2\}$
$+\ 12\{10\ 75431\}$	$+\ 9\{10\ 7542^2\}$	$+\ 9\{10\ 753^22\}$	$+\ 4\{10\ 74^31\}$	$+\ 8\{10\ 74^232\}$
$+\ 4\{10\ 743^3\}$	$+\ \{10\ 6^32\}$	$+\ \{10\ 6^31^2\}$	$+\ 3\{10\ 6^253\}$	$+\ 6\{10\ 6^2521\}$
$+\ 2\{10\ 6^24^2\}$	$+\ 9\{10\ 6^2431\}$	$+\ 7\{10\ 6^242^2\}$	$+\ 6\{10\ 6^23^22\}$	$+\ 3\{10\ 65^24\}$
$+\ 9\{10\ 65^231\}$	$+\ 6\{10\ 65^22^2\}$	$+\ 8\{10\ 654^21\}$	$+\ 16\{10\ 65432\}$	$+\ 5\{10\ 653^3\}$
$+\ 6\{10\ 64^32\}$	$+\ 6\{10\ 64^23^2\}$	$+\ \{10\ 5^4\}$	$+\ 4\{10\ 5^341\}$	$+\ 5\{10\ 5^332\}$
$+\ 6\{10\ 5^24^22\}$	$+\ 5\{10\ 5^243^2\}$	$+\ 4\{10\ 54^33\}$	$+\ \{10\ 4^5\}$	$+\ \{9^2642\}$
$+\ \{9^2641^2\}$	$+\ \{9^263^2\}$	$+\ 2\{9^26321\}$	$+\ \{9^262^3\}$	$+\ \{9^25^22\}$
$+\ \{9^25^21^2\}$	$+\ 2\{9^2543\}$	$+\ 4\{9^25421\}$	$+\ 3\{9^253^21\}$	$+\ 3\{9^2532^2\}$
$+\ \{9^24^3\}$	$+\ 3\{9^24^231\}$	$+\ 2\{9^24^22^2\}$	$+\ 3\{9^243^22\}$	$+\ \{9^23^4\}$
$+\ 2\{98742\}$	$+\ 2\{98741^2\}$	$+\ 2\{9873^2\}$	$+\ 4\{987321\}$	$+\ 2\{9872^3\}$
$+\ 4\{98652\}$	$+\ 4\{98651^2\}$	$+\ 8\{98643\}$	$+\ 16\{986421\}$	$+\ 12\{9863^21\}$
$+\ 12\{98632^2\}$	$+\ 6\{985^23\}$	$+\ 12\{985^221\}$	$+\ 6\{9854^2\}$	$+\ 24\{985431\}$
$+\ 18\{98542^2\}$	$+\ 18\{9853^22\}$	$+\ 8\{984^31\}$	$+\ 16\{984^232\}$	$+\ 8\{9843^3\}$
$+\ 3\{97^252\}$	$+\ 3\{97^251^2\}$	$+\ 6\{97^243\}$	$+\ 12\{97^2421\}$	$+\ 9\{97^23^21\}$
$+\ 9\{97^232^2\}$	$+\ 3\{976^22\}$	$+\ 3\{976^21^2\}$	$+\ 12\{97653\}$	$+\ 24\{976521\}$
$+\ 9\{9764^2\}$	$+\ 39\{976431\}$	$+\ 30\{97642^2\}$	$+\ 27\{9763^22\}$	$+\ 9\{975^24\}$
$+\ 30\{975^231\}$	$+\ 21\{975^22^2\}$	$+\ 27\{9754^21\}$	$+\ 54\{975432\}$	$+\ 18\{9753^3\}$
$+\ 18\{974^32\}$	$+\ 18\{974^23^2\}$	$+\ 4\{96^33\}$	$+\ 8\{96^321\}$	$+\ 8\{96^254\}$
$+\ 27\{96^2531\}$	$+\ 19\{96^252^2\}$	$+\ 19\{96^24^21\}$	$+\ 38\{96^2432\}$	$+\ 11\{96^23^3\}$
$+\ 4\{965^3\}$	$+\ 24\{965^241\}$	$+\ 36\{965^232\}$	$+\ 36\{9654^22\}$	$+\ 32\{96543^2\}$
$+\ 16\{964^33\}$	$+\ 5\{95^41\}$	$+\ 15\{95^342\}$	$+\ 10\{95^33^2\}$	$+\ 15\{95^24^23\}$
$+\ 5\{954^4\}$	$+\ \{8^342\}$	$+\ \{8^341^2\}$	$+\ \{8^33^2\}$	$+\ 2\{8^3321\}$
$+\ \{8^32^3\}$	$+\ 3\{8^2752\}$	$+\ 3\{8^2751^2\}$	$+\ 6\{8^2743\}$	$+\ 12\{8^27421\}$
$+\ 9\{8^273^21\}$	$+\ 9\{8^2732^2\}$	$+\ 2\{8^26^22\}$	$+\ 2\{8^26^21^2\}$	$+\ 9\{8^2653\}$
$+\ 18\{8^26521\}$	$+\ 7\{8^264^2\}$	$+\ 30\{8^26431\}$	$+\ 23\{8^2642^2\}$	$+\ 21\{8^263^22\}$
$+\ 6\{8^25^24\}$	$+\ 21\{8^25^231\}$	$+\ 15\{8^25^22^2\}$	$+\ 19\{8^254^21\}$	$+\ 38\{8^25432\}$
$+\ 13\{8^253^3\}$	$+\ 12\{8^24^32\}$	$+\ 12\{8^24^23^2\}$	$+\ 3\{87^262\}$	$+\ 3\{87^261^2\}$
$+\ 9\{87^253\}$	$+\ 18\{87^2521\}$	$+\ 6\{87^24^2\}$	$+\ 27\{87^2431\}$	$+\ 21\{87^242^2\}$
$+\ 18\{87^23^22\}$	$+\ 8\{876^23\}$	$+\ 16\{876^221\}$	$+\ 16\{87654\}$	$+\ 54\{876531\}$
$+\ 38\{87652^2\}$	$+\ 38\{8764^21\}$	$+\ 76\{876432\}$	$+\ 22\{8763^3\}$	$+\ 5\{875^3\}$

$$
\begin{array}{lllll}
+\,36\{875^241\} & +\,57\{875^232\} & +\,54\{8754^22\} & +\,49\{87543^2\} & +\,20\{874^33\} \\
+\,6\{86^34\} & +\,18\{86^331\} & +\,12\{86^32^2\} & +\,6\{86^25^2\} & +\,36\{86^2541\} \\
+\,54\{86^2532\} & +\,42\{86^24^22\} & +\,36\{86^243^2\} & +\,15\{865^31\} & +\,51\{865^242\} \\
+\,36\{865^23^2\} & +\,45\{8654^23\} & +\,9\{864^4\} & +\,10\{85^42\} & +\,20\{85^343\} \\
+\,10\{85^24^3\} & +\,\{7^42\} & +\,\{7^41^2\} & +\,4\{7^363\} & +\,8\{7^3621\} \\
+\,5\{7^354\} & +\,18\{7^3531\} & +\,13\{7^352^2\} & +\,11\{7^34^21\} & +\,22\{7^3432\} \\
+\,6\{7^33^3\} & +\,6\{7^26^24\} & +\,18\{7^26^231\} & +\,12\{7^26^22^2\} & +\,5\{7^265^2\} \\
+\,32\{7^26541\} & +\,49\{7^26532\} & +\,36\{7^264^22\} & +\,31\{7^2643^2\} & +\,10\{7^25^31\} \\
+\,36\{7^25^242\} & +\,26\{7^25^23^2\} & +\,30\{7^254^23\} & +\,5\{7^24^4\} & +\,4\{76^35\} \\
+\,16\{76^341\} & +\,20\{76^332\} & +\,15\{76^25^21\} & +\,45\{76^2542\} & +\,30\{76^253^2\} \\
+\,29\{76^24^23\} & +\,20\{765^32\} & +\,40\{765^243\} & +\,16\{7654^3\} & +\,10\{75^43\} \\
+\,10\{75^34^2\} & +\,\{6^5\} & +\,5\{6^451\} & +\,9\{6^442\} & +\,5\{6^43^2\} \\
+\,10\{6^35^22\} & +\,16\{6^3543\} & +\,5\{6^34^3\} & +\,10\{6^25^33\} & +\,9\{6^25^24^2\} \\
+\,5\{65^44\} & +\,\{5^6\} & & &
\end{array}
$$

17. The case of $q = 0$

When $q = 0$ we have

$$\Delta_N^2 = (-1)^{[N/2]}\{(N-1)^N\} \tag{17-1}$$

18. The case of $q^2 + q + 1 = 0$

We note that the factor $q^2 + q + 1 = 0$ occurs in all of the $q-$polynomials apart from one case for each value of N, namely the term

$$q^{\frac{N(N-1)}{2}}\{2N-2, 2N-4, \ldots, 2\} \tag{17-2}$$

Furthermore, in U_N we have

$$dim\{2N-2, 2N-4, \ldots, 2\} = 3^{N(N-1)/2} \tag{17-2$'$}$$

We further note that for $q = 1$ all multiplicities are divisible by 3 apart from the case of (17-2) which is multiplicity free.

19. The case of $q = \sqrt{-1}$

Consider the expansion

$$\Delta_N = \prod_{i<j}^{N}(x_i - \sqrt{-1}x_j)(\sqrt{-1}x_i - x_j) \tag{18-1}$$

Clearly

$$\Delta_2 = \sqrt{-1}(x_1^2 + x_2^2) = \sqrt{-1}(\{2\} - \{1^2\}) \tag{18-2}$$

For general N we suggest

$$\Delta_N = (\sqrt{-1})^{\frac{N(N-1)}{2}}(\{2\} - \{1^2\}) \otimes \{N-1, N-2, \ldots, 1\} \tag{18-3}$$

evaluated in $U(N)$. Since $\{2\} - \{1^2\}$ is just the N component monomial function m_2 the required plethysm may be evaluated by expanding $\{N-1, N-2, \ldots, 1\}$ as a sum of monomials of length $\leq N$, doubling their parts and then expanding the resultant monomials as a sum of $S-$functions of length $\leq N$. Thus we obtain

$$\begin{array}{ll} N & \Delta_N \\ 2 & \sqrt{-1}(\{2\} - \{1^2\}) \\ 3 & -\sqrt{-1}(\{42\} - \{41^2\} - \{3^2\} + \{2^3\}) \\ 4 & \{642\} - \{641^2\} - \{63^2\} + \{62^3\} - \{5^22\} + \{5^21^2\} + \{53^21\} \\ & -\{532^2\} + \{4^3\} - \{4^231\} + 2\{4^22^2\} - \{43^22\} + \{3^4\} \end{array} \tag{18-4}$$

and for $N = 5$ we have

$$\begin{array}{lllll} -\{8642\} & +\{8641^2\} & +\{863^2\} & -\{862^3\} & +\{85^22\} \\ -\{85^21^2\} & -\{853^21\} & +\{8532^2\} & -\{84^3\} & +\{84^231\} \\ -2\{84^22^2\} & +\{843^22\} & -\{83^4\} & +\{7^242\} & -\{7^241^2\} \\ -\{7^23^2\} & +\{7^22^3\} & -\{75^23\} & +\{754^2\} & +\{7542^2\} \\ -\{753^22\} & -\{6^32\} & +\{6^31^2\} & +\{6^253\} & -2\{6^24^2\} \\ +\{6^2431\} & -3\{6^242^2\} & +2\{6^23^22\} & +\{65^24\} & -\{65^231\} \\ +2\{65^22^2\} & -\{653^3\} & -2\{64^32\} & +2\{64^23^2\} & -\{5^4\} \\ -\{5^332\} & +2\{5^24^22\} & -\{5^243^2\} & -\{4^5\} & \end{array}$$

and for $N = 6$

$\{10\ 8642\}$	$-\ \{10\ 8641^{2}\}$	$-\ \{10\ 863^{2}\}$	$+\ \{10\ 862^{3}\}$	$-\ \{10\ 85^{2}2\}$
$+\ \{10\ 85^{2}1^{2}\}$	$+\ \{10\ 853^{2}1\}$	$-\ \{10\ 8532^{2}\}$	$+\ \{10\ 84^{3}\}$	$-\ \{10\ 84^{2}31\}$
$+\ 2\{10\ 84^{2}2^{2}\}$	$-\ \{10\ 843^{2}2\}$	$+\ \{10\ 83^{4}\}$	$-\ \{10\ 7^{2}42\}$	$+\ \{10\ 7^{2}41^{2}\}$
$+\ \{10\ 7^{2}3^{2}\}$	$-\ \{10\ 7^{2}2^{3}\}$	$+\ \{10\ 75^{2}3\}$	$-\ \{10\ 754^{2}\}$	$-\ \{10\ 7542^{2}\}$
$+\ \{10\ 753^{2}2\}$	$+\ \{10\ 6^{3}2\}$	$-\ \{10\ 6^{3}1^{2}\}$	$-\ \{10\ 6^{2}53\}$	$+\ 2\{10\ 6^{2}4^{2}\}$
$-\ \{10\ 6^{2}431\}$	$+\ 3\{10\ 6^{2}42^{2}\}$	$-\ 2\{10\ 6^{2}3^{2}2\}$	$-\ \{10\ 65^{2}4\}$	$+\ \{10\ 65^{2}31\}$
$-\ 2\{10\ 65^{2}2^{2}\}$	$+\ \{10\ 653^{3}\}$	$+\ 2\{10\ 64^{3}2\}$	$-\ 2\{10\ 64^{2}3^{2}\}$	$+\ \{10\ 5^{4}\}$
$+\ \{10\ 5^{3}32\}$	$-\ 2\{10\ 5^{2}4^{2}2\}$	$+\ \{10\ 5^{2}43^{2}\}$	$+\ \{10\ 4^{5}\}$	$-\ \{9^{2}642\}$
$+\ \{9^{2}641^{2}\}$	$+\ \{9^{2}63^{2}\}$	$-\ \{9^{2}62^{3}\}$	$+\ \{9^{2}5^{2}2\}$	$-\ \{9^{2}5^{2}1^{2}\}$
$-\ \{9^{2}53^{2}1\}$	$+\ \{9^{2}532^{2}\}$	$-\ \{9^{2}4^{3}\}$	$+\ \{9^{2}4^{2}31\}$	$-\ 2\{9^{2}4^{2}2^{2}\}$
$+\ \{9^{2}43^{2}2\}$	$-\ \{9^{2}3^{4}\}$	$+\ \{97^{2}52\}$	$-\ \{97^{2}51^{2}\}$	$-\ \{97^{2}3^{2}1\}$
$+\ \{97^{2}32^{2}\}$	$-\ \{976^{2}2\}$	$+\ \{976^{2}1^{2}\}$	$-\ \{9764^{2}\}$	$+\ \{976431\}$
$-\ 2\{97642^{2}\}$	$+\ \{9763^{2}2\}$	$+\ \{975^{2}4\}$	$-\ 2\{975^{2}31\}$	$+\ 3\{975^{2}2^{2}\}$
$+\ \{9754^{2}1\}$	$-\ 2\{9753^{3}\}$	$-\ 2\{974^{3}2\}$	$+\ 2\{974^{2}3^{2}\}$	$+\ \{96^{2}531\}$
$-\ \{96^{2}52^{2}\}$	$-\ \{96^{2}4^{2}1\}$	$+\ \{96^{2}3^{3}\}$	$-\ \{95^{4}1\}$	$+\ \{95^{3}42\}$
$-\ 2\{95^{3}3^{2}\}$	$+\ \{95^{2}4^{2}3\}$	$-\ \{954^{4}\}$	$+\ \{8^{3}42\}$	$-\ \{8^{3}41^{2}\}$
$-\ \{8^{3}3^{2}\}$	$+\ \{8^{3}2^{3}\}$	$-\ \{8^{2}752\}$	$+\ \{8^{2}751^{2}\}$	$+\ \{8^{2}73^{2}1\}$
$-\ \{8^{2}732^{2}\}$	$+\ 2\{8^{2}6^{2}2\}$	$-\ 2\{8^{2}6^{2}1^{2}\}$	$-\ \{8^{2}653\}$	$+\ 3\{8^{2}64^{2}\}$
$-\ 2\{8^{2}6431\}$	$+\ 5\{8^{2}642^{2}\}$	$-\ 3\{8^{2}63^{2}2\}$	$-\ 2\{8^{2}5^{2}4\}$	$+\ 3\{8^{2}5^{2}31\}$
$-\ 5\{8^{2}5^{2}2^{2}\}$	$-\ \{8^{2}54^{2}1\}$	$+\ 3\{8^{2}53^{3}\}$	$+\ 4\{8^{2}4^{3}2\}$	$-\ 4\{8^{2}4^{2}3^{2}\}$
$-\ \{87^{2}62\}$	$+\ \{87^{2}61^{2}\}$	$+\ \{87^{2}53\}$	$-\ 2\{87^{2}4^{2}\}$	$+\ \{87^{2}431\}$
$-\ 3\{87^{2}42^{2}\}$	$+\ 2\{87^{2}3^{2}2\}$	$+\ \{875^{3}\}$	$+\ \{875^{2}32\}$	$-\ 2\{8754^{2}2\}$
$+\ \{87543^{2}\}$	$+\ 2\{86^{3}4\}$	$-\ 2\{86^{3}31\}$	$+\ 4\{86^{3}2^{2}\}$	$-\ 2\{86^{2}5^{2}\}$
$-\ 2\{86^{2}532\}$	$+\ 6\{86^{2}4^{2}2\}$	$-\ 4\{86^{2}43^{2}\}$	$+\ \{865^{3}1\}$	$-\ 3\{865^{2}42\}$
$+\ 4\{865^{2}3^{2}\}$	$-\ \{8654^{2}3\}$	$+\ 3\{864^{4}\}$	$+\ 2\{85^{4}2\}$	$-\ 2\{85^{2}4^{3}\}$
$+\ \{7^{4}2\}$	$-\ \{7^{4}1^{2}\}$	$+\ \{7^{3}54\}$	$-\ 2\{7^{3}531\}$	$+\ 3\{7^{3}52^{2}\}$
$+\ \{7^{3}4^{2}1\}$	$-\ 2\{7^{3}3^{3}\}$	$-\ 2\{7^{2}6^{2}4\}$	$+\ 2\{7^{2}6^{2}31\}$	$-\ 4\{7^{2}6^{2}2^{2}\}$
$+\ \{7^{2}65^{2}\}$	$+\ \{7^{2}6532\}$	$-\ 4\{7^{2}64^{2}2\}$	$+\ 3\{7^{2}643^{2}\}$	$-\ 2\{7^{2}5^{3}1\}$
$+\ 4\{7^{2}5^{2}42\}$	$-\ 6\{7^{2}5^{2}3^{2}\}$	$+\ 2\{7^{2}54^{2}3\}$	$-\ 3\{7^{2}4^{4}\}$	$+\ \{76^{2}5^{2}1\}$
$-\ \{76^{2}542\}$	$+\ 2\{76^{2}53^{2}\}$	$-\ \{76^{2}4^{2}3\}$	$-\ 2\{75^{4}3\}$	$+\ 2\{75^{3}4^{2}\}$
$+\ \{6^{5}\}$	$-\ \{6^{4}51\}$	$+\ 3\{6^{4}42\}$	$-\ 3\{6^{4}3^{2}\}$	$-\ 2\{6^{3}5^{2}2\}$
$+\ 3\{6^{3}4^{3}\}$	$+\ 2\{6^{2}5^{3}3\}$	$-\ 3\{6^{2}5^{2}4^{2}\}$	$+\ \{65^{4}4\}$	$-\ \{5^{6}\}$

and for $N = 7$

$$
\begin{array}{lllll}
\{12\;10\;8642\} & -\{12\;10\;8641^{2}\} & -\{12\;10\;863^{2}\} & +\{12\;10\;862^{3}\} & -\{12\;10\;85^{2}2\} \\
+\{12\;10\;85^{2}1^{2}\} & +\{12\;10\;853^{2}1\} & -\{12\;10\;8532^{2}\} & +\{12\;10\;84^{3}\} & -\{12\;10\;84^{2}31\} \\
+2\{12\;10\;84^{2}2^{2}\} & -\{12\;10\;843^{2}2\} & +\{12\;10\;83^{4}\} & -\{12\;10\;7^{2}42\} & +\{12\;10\;7^{2}41^{2}\} \\
+\{12\;10\;7^{2}3^{2}\} & -\{12\;10\;7^{2}2^{3}\} & +\{12\;10\;75^{2}3\} & -\{12\;10\;754^{2}\} & -\{12\;10\;7542^{2}\} \\
+\{12\;10\;753^{2}2\} & +\{12\;10\;6^{3}2\} & -\{12\;10\;6^{3}1^{2}\} & -\{12\;10\;6^{2}53\} & +2\{12\;10\;6^{2}4^{2}\} \\
-\{12\;10\;6^{2}431\} & +3\{12\;10\;6^{2}42^{2}\} & -2\{12\;10\;6^{2}3^{2}2\} & -\{12\;10\;65^{2}4\} & +\{12\;10\;65^{2}31\} \\
-2\{12\;10\;65^{2}2^{2}\} & +\{12\;10\;653^{3}\} & +2\{12\;10\;64^{3}2\} & -2\{12\;10\;64^{2}3^{2}\} & +\{12\;10\;5^{4}\} \\
+\{12\;10\;5^{3}32\} & -2\{12\;10\;5^{2}4^{2}2\} & +\{12\;10\;5^{2}43^{2}\} & +\{12\;10\;4^{5}\} & -\{12\;9^{2}642\} \\
+\{12\;9^{2}641^{2}\} & +\{12\;9^{2}63^{2}\} & -\{12\;9^{2}62^{3}\} & +\{12\;9^{2}5^{2}2\} & -\{12\;9^{2}5^{2}1^{2}\} \\
-\{12\;9^{2}53^{2}1\} & +\{12\;9^{2}532^{2}\} & -\{12\;9^{2}4^{3}\} & +\{12\;9^{2}4^{2}31\} & -2\{12\;9^{2}4^{2}2^{2}\} \\
+\{12\;9^{2}43^{2}2\} & -\{12\;9^{2}3^{4}\} & +\{12\;97^{2}52\} & -\{12\;97^{2}51^{2}\} & -\{12\;97^{2}3^{2}1\} \\
+\{12\;97^{2}32^{2}\} & -\{12\;976^{2}2\} & +\{12\;976^{2}1^{2}\} & -\{12\;9764^{2}\} & +\{12\;976431\} \\
-2\{12\;97642^{2}\} & +\{12\;9763^{2}2\} & +\{12\;975^{2}4\} & -2\{12\;975^{2}31\} & +3\{12\;975^{2}2^{2}\} \\
+\{12\;9754^{2}1\} & -2\{12\;9753^{3}\} & -2\{12\;974^{3}2\} & +2\{12\;974^{2}3^{2}\} & +\{12\;96^{2}531\} \\
-\{12\;96^{2}52^{2}\} & -\{12\;96^{2}4^{2}1\} & +\{12\;96^{2}3^{3}\} & -\{12\;95^{4}1\} & +\{12\;95^{3}42\} \\
-2\{12\;95^{3}3^{2}\} & +\{12\;95^{2}4^{2}3\} & -\{12\;954^{4}\} & +\{12\;8^{3}42\} & -\{12\;8^{3}41^{2}\} \\
-\{12\;8^{3}3^{2}\} & +\{12\;8^{3}2^{3}\} & -\{12\;8^{2}752\} & +\{12\;8^{2}751^{2}\} & +\{12\;8^{2}73^{2}1\} \\
-\{12\;8^{2}732^{2}\} & +2\{12\;8^{2}6^{2}2\} & -2\{12\;8^{2}6^{2}1^{2}\} & -\{12\;8^{2}653\} & +3\{12\;8^{2}64^{2}\} \\
-2\{12\;8^{2}6431\} & +5\{12\;8^{2}642^{2}\} & -3\{12\;8^{2}63^{2}2\} & -2\{12\;8^{2}5^{2}4\} & +3\{12\;8^{2}5^{2}31\} \\
-5\{12\;8^{2}5^{2}2^{2}\} & -\{12\;8^{2}54^{2}1\} & +3\{12\;8^{2}53^{3}\} & +4\{12\;8^{2}4^{3}2\} & -4\{12\;8^{2}4^{2}3^{2}\} \\
-\{12\;87^{2}62\} & +\{12\;87^{2}61^{2}\} & +\{12\;87^{2}53\} & -2\{12\;87^{2}4^{2}\} & +\{12\;87^{2}431\} \\
-3\{12\;87^{2}42^{2}\} & +2\{12\;87^{2}3^{2}2\} & +\{12\;875^{3}\} & +\{12\;875^{2}32\} & -2\{12\;8754^{2}2\} \\
+\{12\;87543^{2}\} & +2\{12\;86^{3}4\} & -2\{12\;86^{3}31\} & +4\{12\;86^{3}2^{2}\} & -2\{12\;86^{2}5^{2}\} \\
-2\{12\;86^{2}532\} & +6\{12\;86^{2}4^{2}2\} & -4\{12\;86^{2}43^{2}\} & +\{12\;865^{3}1\} & -3\{12\;865^{2}42\} \\
+4\{12\;865^{2}3^{2}\} & -\{12\;8654^{2}3\} & +3\{12\;864^{4}\} & +2\{12\;85^{4}2\} & -2\{12\;85^{2}4^{3}\} \\
+\{12\;7^{4}2\} & -\{12\;7^{4}1^{2}\} & +\{12\;7^{3}54\} & -2\{12\;7^{3}531\} & +3\{12\;7^{3}52^{2}\} \\
+\{12\;7^{3}4^{2}1\} & -2\{12\;7^{3}3^{3}\} & -2\{12\;7^{2}6^{2}4\} & +2\{12\;7^{2}6^{2}31\} & -4\{12\;7^{2}6^{2}2^{2}\} \\
+\{12\;7^{2}65^{2}\} & +\{12\;7^{2}6532\} & -4\{12\;7^{2}64^{2}2\} & +3\{12\;7^{2}643^{2}\} & -2\{12\;7^{2}5^{3}1\} \\
+4\{12\;7^{2}5^{2}42\} & -6\{12\;7^{2}5^{2}3^{2}\} & +2\{12\;7^{2}54^{2}3\} & -3\{12\;7^{2}4^{4}\} & +\{12\;76^{2}5^{2}1\} \\
-\{12\;76^{2}542\} & +2\{12\;76^{2}53^{2}\} & -\{12\;76^{2}4^{2}3\} & -2\{12\;75^{4}3\} & +2\{12\;75^{3}4^{2}\} \\
+\{12\;6^{5}\} & -\{12\;6^{4}51\} & +3\{12\;6^{4}42\} & -3\{12\;6^{4}3^{2}\} & -2\{12\;6^{3}5^{2}2\} \\
+3\{12\;6^{3}4^{3}\} & +2\{12\;6^{2}5^{3}3\} & -3\{12\;6^{2}5^{2}4^{2}\} & +\{12\;65^{4}4\} & -\{12\;5^{6}\} \\
-\{11^{2}8642\} & +\{11^{2}8641^{2}\} & +\{11^{2}863^{2}\} & -\{11^{2}862^{3}\} & +\{11^{2}85^{2}2\} \\
-\{11^{2}85^{2}1^{2}\} & -\{11^{2}853^{2}1\} & +\{11^{2}8532^{2}\} & -\{11^{2}84^{3}\} & +\{11^{2}84^{2}31\} \\
-2\{11^{2}84^{2}2^{2}\} & +\{11^{2}843^{2}2\} & -\{11^{2}83^{4}\} & +\{11^{2}7^{2}42\} & -\{11^{2}7^{2}41^{2}\} \\
-\{11^{2}7^{2}3^{2}\} & +\{11^{2}7^{2}2^{3}\} & -\{11^{2}75^{2}3\} & +\{11^{2}754^{2}\} & +\{11^{2}7542^{2}\} \\
-\{11^{2}753^{2}2\} & -\{11^{2}6^{3}2\} & +\{11^{2}6^{3}1^{2}\} & +\{11^{2}6^{2}53\} & -2\{11^{2}6^{2}4^{2}\}
\end{array}
$$

$+\{11^2 6^2 431\} \quad -3\{11^2 6^2 42^2\} \quad +2\{11^2 6^2 3^2 2\} \quad +\{11^2 65^2 4\} \quad -\{11^2 65^2 31\}$

$+2\{11^2 65^2 2^2\} \quad -\{11^2 653^3\} \quad -2\{11^2 64^3 2\} \quad +2\{11^2 64^2 3^2\} \quad -\{11^2 5^4\}$

$-\{11^2 5^3 32\} \quad +2\{11^2 5^2 4^2 2\} \quad -\{11^2 5^2 43^2\} \quad -\{11^2 4^5\} \quad +\{11\ 9^2 742\}$

$-\{11\ 9^2 741^2\} \quad -\{11\ 9^2 73^2\} \quad +\{11\ 9^2 72^3\} \quad -\{11\ 9^2 5^2 3\} \quad +\{11\ 9^2 54^2\}$

$+\{11\ 9^2 542^2\} \quad -\{11\ 9^2 53^2 2\} \quad -\{11\ 98^2 42\} \quad +\{11\ 98^2 41^2\} \quad +\{11\ 98^2 3^2\}$

$-\{11\ 98^2 2^3\} \quad -\{11\ 986^2 2\} \quad +\{11\ 986^2 1^2\} \quad +\{11\ 98653\} \quad -2\{11\ 9864^2\}$

$+\{11\ 986431\} \quad -3\{11\ 98642^2\} \quad +2\{11\ 9863^2 2\} \quad +\{11\ 985^2 4\} \quad -\{11\ 985^2 31\}$

$+2\{11\ 985^2 2^2\} \quad -\{11\ 9853^3\} \quad -2\{11\ 984^3 2\} \quad +2\{11\ 984^2 3^2\} \quad +\{11\ 97^2 62\}$

$-\{11\ 97^2 61^2\} \quad -2\{11\ 97^2 53\} \quad +3\{11\ 97^2 4^2\} \quad -\{11\ 97^2 431\} \quad +4\{11\ 97^2 42^2\}$

$-3\{11\ 97^2 3^2 2\} \quad +\{11\ 976^2 3\} \quad -2\{11\ 975^3\} \quad -2\{11\ 975^2 32\} \quad +3\{11\ 9754^2 2\}$

$-\{11\ 97543^2\} \quad -2\{11\ 96^3 4\} \quad +\{11\ 96^3 31\} \quad -3\{11\ 96^3 2^2\} \quad +2\{11\ 96^2 5^2\}$

$+2\{11\ 96^2 532\} \quad -5\{11\ 96^2 4^2 2\} \quad +3\{11\ 96^2 43^2\} \quad +2\{11\ 965^2 42\} \quad -2\{11\ 965^2 3^2\}$

$-2\{11\ 964^4\} \quad -2\{11\ 95^4 2\} \quad +2\{11\ 95^2 4^3\} \quad +\{11\ 8^2 753\} \quad -\{11\ 8^2 74^2\}$

$-\{11\ 8^2 742^2\} \quad +\{11\ 8^2 73^2 2\} \quad -\{11\ 8^2 6^2 3\} \quad +\{11\ 8^2 5^3\} \quad +\{11\ 8^2 5^2 32\}$

$-\{11\ 8^2 54^2 2\} \quad -\{11\ 7^4 3\} \quad +\{11\ 7^3 64\} \quad +\{11\ 7^3 62^2\} \quad -2\{11\ 7^3 5^2\}$

$-2\{11\ 7^3 532\} \quad +3\{11\ 7^3 4^2 2\} \quad -\{11\ 7^3 43^2\} \quad +\{11\ 7^2 6^2 5\} \quad +\{11\ 7^2 6^2 32\}$

$-2\{11\ 7^2 5^3 2\} \quad +\{11\ 7^2 54^3\} \quad -\{11\ 76^4\} \quad -2\{11\ 76^3 42\} \quad +\{11\ 76^3 3^2\}$

$+2\{11\ 76^2 5^2 2\} \quad -2\{11\ 76^2 4^3\} \quad +\{11\ 765^2 4^2\} \quad -\{11\ 75^4 4\} \quad +\{10^3 642\}$

$-\{10^3 641^2\} \quad -\{10^3 63^2\} \quad +\{10^3 62^3\} \quad -\{10^3 5^2 2\} \quad +\{10^3 5^2 1^2\}$

$+\{10^3 53^2 1\} \quad -\{10^3 532^2\} \quad +\{10^3 4^3\} \quad -\{10^3 4^2 31\} \quad +2\{10^3 4^2 2^2\}$

$-\{10^3 43^2 2\} \quad +\{10^3 3^4\} \quad -\{10^2 9742\} \quad +\{10^2 9741^2\} \quad +\{10^2 973^2\}$

$-\{10^2 972^3\} \quad +\{10^2 95^2 3\} \quad -\{10^2 954^2\} \quad -\{10^2 9542^2\} \quad +\{10^2 953^2 2\}$

$+2\{10^2 8^2 42\} \quad -2\{10^2 8^2 41^2\} \quad -2\{10^2 8^2 3^2\} \quad +2\{10^2 8^2 2^3\} \quad -\{10^2 8752\}$

$+\{10^2 8751^2\} \quad +\{10^2 873^2 1\} \quad -\{10^2 8732^2\} \quad +3\{10^2 86^2 2\} \quad -3\{10^2 86^2 1^2\}$

$-2\{10^2 8653\} \quad +5\{10^2 864^2\} \quad -3\{10^2 86431\} \quad +8\{10^2 8642^2\} \quad -5\{10^2 863^2 2\}$

$-3\{10^2 85^2 4\} \quad +4\{10^2 85^2 31\} \quad -7\{10^2 85^2 2^2\} \quad -\{10^2 854^2 1\} \quad +4\{10^2 853^3\}$

$+6\{10^2 84^3 2\} \quad -6\{10^2 84^2 3^2\} \quad -2\{10^2 7^2 62\} \quad +2\{10^2 7^2 61^2\} \quad +3\{10^2 7^2 53\}$

$-5\{10^2 7^2 4^2\} \quad +2\{10^2 7^2 431\} \quad -7\{10^2 7^2 42^2\} \quad +5\{10^2 7^2 3^2 2\} \quad -\{10^2 76^2 3\}$

$+3\{10^2 75^3\} \quad +3\{10^2 75^2 32\} \quad -5\{10^2 754^2 2\} \quad +2\{10^2 7543^2\} \quad +4\{10^2 6^3 4\}$

$-3\{10^2 6^3 31\} \quad +7\{10^2 6^3 2^2\} \quad -4\{10^2 6^2 5^2\} \quad -4\{10^2 6^2 532\} \quad +11\{10^2 6^2 4^2 2\}$

$-7\{10^2 6^2 43^2\} \quad +\{10^2 65^3 1\} \quad -5\{10^2 65^2 42\} \quad +6\{10^2 65^2 3^2\} \quad -\{10^2 654^2 3\}$

$+5\{10^2 64^4\} \quad +4\{10^2 5^4 2\} \quad -4\{10^2 5^2 4^3\} \quad -\{10\ 9^2 842\} \quad +\{10\ 9^2 841^2\}$

$+\{10\ 9^2 83^2\} \quad -\{10\ 9^2 82^3\} \quad +\{10\ 9^2 752\} \quad -\{10\ 9^2 751^2\} \quad -\{10\ 9^2 73^2 1\}$

$+\{10\ 9^2 732^2\} \quad -2\{10\ 9^2 6^2 2\} \quad +2\{10\ 9^2 6^2 1^2\} \quad +\{10\ 9^2 653\} \quad -3\{10\ 9^2 64^2\}$

$+2\{10\ 9^2 6431\} \quad -5\{10\ 9^2 642^2\} \quad +3\{10\ 9^2 63^2 2\} \quad +2\{10\ 9^2 5^2 4\} \quad -3\{10\ 9^2 5^2 31\}$

$+5\{10\ 9^2 5^2 2^2\} \quad +\{10\ 9^2 54^2 1\} \quad -3\{10\ 9^2 53^3\} \quad -4\{10\ 9^2 4^3 2\} \quad +4\{10\ 9^2 4^2 3^2\}$

$+\{10\ 97^3 2\} \quad -\{10\ 97^3 1^2\} \quad +\{10\ 97^2 54\} \quad -2\{10\ 97^2 531\} \quad +3\{10\ 97^2 52^2\}$

$+\{10\ 97^2 4^2 1\} \quad -2\{10\ 97^2 3^3\} \quad -2\{10\ 976^2 4\} \quad +2\{10\ 976^2 31\} \quad -4\{10\ 976^2 2^2\}$

$+\{10\ 9765^2\} \quad +\{10\ 976532\} \quad -4\{10\ 9764^2 2\} \quad +3\{10\ 97643^2\} \quad -2\{10\ 975^3 1\}$

$+4\{10\ 975^2 42\} \quad -6\{10\ 975^2 3^2\} \quad +2\{10\ 9754^2 3\} \quad -3\{10\ 974^4\} \quad +\{10\ 96^2 5^2 1\}$

$- \{10\ 96^2542\} \quad + 2\{10\ 96^253^2\} \quad - \{10\ 96^24^23\} \quad - 2\{10\ 95^43\} \quad + 2\{10\ 95^34^2\}$
$+ 2\{10\ 8^362\} \quad - 2\{10\ 8^361^2\} \quad - 2\{10\ 8^353\} \quad + 4\{10\ 8^34^2\} \quad - 2\{10\ 8^3431\}$
$+ 6\{10\ 8^342^2\} \quad - 4\{10\ 8^33^22\} \quad - 2\{10\ 8^27^22\} \quad + 2\{10\ 8^27^21^2\} \quad - 2\{10\ 8^2754\}$
$+ 3\{10\ 8^27531\} \quad - 5\{10\ 8^2752^2\} \quad - \{10\ 8^274^21\} \quad + 3\{10\ 8^273^3\} \quad + 6\{10\ 8^26^24\}$
$- 5\{10\ 8^26^231\} \quad + 11\{10\ 8^26^22^2\} \quad - 4\{10\ 8^265^2\} \quad - 4\{10\ 8^26532\} \quad + 13\{10\ 8^264^22\}$
$- 9\{10\ 8^2643^2\} \quad + 3\{10\ 8^25^31\} \quad - 9\{10\ 8^25^242\} \quad + 12\{10\ 8^25^23^2\} \quad - 3\{10\ 8^254^23\}$
$+ 7\{10\ 8^24^4\} \quad + \{10\ 87^33\} \quad - 3\{10\ 87^264\} \quad + 2\{10\ 87^2631\} \quad - 5\{10\ 87^262^2\}$
$+ 4\{10\ 87^25^2\} \quad + 4\{10\ 87^2532\} \quad - 9\{10\ 87^24^22\} \quad + 5\{10\ 87^243^2\} \quad - \{10\ 876^25\}$
$- \{10\ 876^232\} \quad + 4\{10\ 875^32\} \quad - 3\{10\ 8754^3\} \quad + 3\{10\ 86^4\} \quad - 2\{10\ 86^351\}$
$+ 10\{10\ 86^342\} \quad - 9\{10\ 86^33^2\} \quad - 8\{10\ 86^25^22\} \quad + 10\{10\ 86^24^3\} \quad + 4\{10\ 865^33\}$
$- 7\{10\ 865^24^2\} \quad + 3\{10\ 85^44\} \quad + 2\{10\ 7^44\} \quad - 2\{10\ 7^431\} \quad + 4\{10\ 7^42^2\}$
$- 2\{10\ 7^35^21\} \quad + 4\{10\ 7^3542\} \quad - 6\{10\ 7^353^2\} \quad + 2\{10\ 7^34^23\} \quad - 2\{10\ 7^26^3\}$
$+ 2\{10\ 7^26^251\} \quad - 8\{10\ 7^26^242\} \quad + 8\{10\ 7^26^23^2\} \quad + 4\{10\ 7^265^22\} \quad - 6\{10\ 7^264^3\}$
$- 6\{10\ 7^25^33\} \quad + 8\{10\ 7^25^24^2\} \quad + 2\{10\ 76^25^23\} \quad - 2\{10\ 76^254^2\} \quad - 2\{10\ 75^5\}$
$+ 3\{10\ 6^52\} \quad - 3\{10\ 6^453\} \quad + 6\{10\ 6^44^2\} \quad - 3\{10\ 6^35^24\} \quad + 3\{10\ 6^25^4\}$
$+ \{9^442\} \quad - \{9^441^2\} \quad - \{9^43^2\} \quad + \{9^42^3\} \quad + \{9^3762\}$
$- \{9^3761^2\} \quad - 2\{9^3753\} \quad + 3\{9^374^2\} \quad - \{9^37431\} \quad + 4\{9^3742^2\}$
$- 3\{9^373^22\} \quad + \{9^36^23\} \quad - 2\{9^35^3\} \quad - 2\{9^35^232\} \quad + 3\{9^354^22\}$
$- \{9^3543^2\} \quad - 2\{9^28^262\} \quad + 2\{9^28^261^2\} \quad + 2\{9^28^253\} \quad - 4\{9^28^24^2\}$
$+ 2\{9^28^2431\} \quad - 6\{9^28^242^2\} \quad + 4\{9^28^23^22\} \quad + \{9^287^22\} \quad - \{9^287^21^2\}$
$+ \{9^28754\} \quad - \{9^287531\} \quad + 2\{9^28752^2\} \quad - \{9^2873^3\} \quad - 4\{9^286^24\}$
$+ 3\{9^286^231\} \quad - 7\{9^286^22^2\} \quad + 3\{9^2865^2\} \quad + 3\{9^286532\} \quad - 9\{9^2864^22\}$
$+ 6\{9^28643^2\} \quad - \{9^285^31\} \quad + 5\{9^285^242\} \quad - 6\{9^285^23^2\} \quad + \{9^2854^23\}$
$- 4\{9^284^4\} \quad - 2\{9^27^33\} \quad + 4\{9^27^264\} \quad - 2\{9^27^2631\} \quad + 6\{9^27^262^2\}$
$- 6\{9^27^25^2\} \quad - 6\{9^27^2532\} \quad + 12\{9^27^24^22\} \quad - 6\{9^27^243^2\} \quad + 2\{9^276^25\}$
$+ 2\{9^276^232\} \quad - 6\{9^275^32\} \quad + 4\{9^2754^3\} \quad - 3\{9^26^4\} \quad + \{9^26^351\}$
$- 9\{9^26^342\} \quad + 7\{9^26^33^2\} \quad + 8\{9^26^25^22\} \quad - 9\{9^26^24^3\} \quad - 2\{9^265^33\}$
$+ 5\{9^265^24^2\} \quad - 3\{9^25^44\} \quad + \{98^27^23\} \quad - \{98^2764\} \quad - \{98^2762^2\}$
$+ 2\{98^275^2\} \quad + 2\{98^27532\} \quad - 3\{98^274^22\} \quad + \{98^2743^2\} \quad - \{98^26^25\}$
$- \{98^26^232\} \quad + 2\{98^25^32\} \quad - \{98^254^3\} \quad - 2\{97^45\} \quad - 2\{97^432\}$
$+ 2\{97^36^2\} \quad + 4\{97^3642\} \quad - 2\{97^363^2\} \quad - 6\{97^35^22\} \quad + 4\{97^34^3\}$
$+ 2\{97^26^252\} \quad - 2\{97^25^34\} \quad - 3\{976^42\} \quad + \{976^353\} \quad - 4\{976^34^2\}$
$+ 3\{976^25^24\} \quad - \{9765^4\} \quad + \{8^52\} \quad - \{8^51^2\} \quad - \{8^473\}$
$+ 3\{8^464\} \quad - 2\{8^4631\} \quad + 5\{8^462^2\} \quad - 3\{8^45^2\} \quad - 3\{8^4532\}$
$+ 7\{8^44^22\} \quad - 4\{8^443^2\} \quad - 2\{8^37^24\} \quad + 2\{8^37^231\} \quad - 4\{8^37^22^2\}$
$+ \{8^375^21\} \quad - 3\{8^37542\} \quad + 4\{8^3753^2\} \quad - \{8^374^23\} \quad + 3\{8^36^3\}$
$- 2\{8^36^251\} \quad + 10\{8^36^242\} \quad - 9\{8^36^23^2\} \quad - 6\{8^365^22\} \quad + 8\{8^364^3\}$
$+ 4\{8^35^33\} \quad - 7\{8^35^24^2\} \quad + 2\{8^27^35\} \quad + 2\{8^27^332\} \quad - 3\{8^27^26^2\}$
$+ \{8^27^2651\} \quad - 7\{8^27^2642\} \quad + 5\{8^27^263^2\} \quad + 8\{8^27^25^22\} \quad - 7\{8^27^24^3\}$
$- 2\{8^276^252\} \quad + 3\{8^275^34\} \quad + 6\{8^26^42\} \quad - 4\{8^26^353\} \quad + 10\{8^26^34^2\}$
$- 6\{8^26^25^24\} \quad + 3\{8^265^4\} \quad + \{87^46\} \quad - \{87^451\} \quad + 3\{87^442\}$

$-\ 3\{874^43^2\}$	$-\ 2\{873^35^23\}$	$+\ 3\{873^354^2\}$	$-\ 3\{872^26^32\}$	$+\ 3\{872^26^253\}$
$-\ 6\{872^26^24^2\}$	$+\ 2\{872^265^24\}$	$-\ 3\{872^25^4\}$	$+\ \{876^25^3\}$	$+\ 3\{86^54\}$
$-\ 3\{86^45^2\}$	$-\ \{7^6\}$	$-\ 2\{7^552\}$	$+\ 3\{7^46^22\}$	$-\ \{7^4653\}$
$+\ 3\{7^464^2\}$	$-\ 3\{7^45^24\}$	$+\ \{7^36^254\}$	$-\ 3\{7^26^44\}$	$+\ 2\{7^26^35^2\}$
$+\ \{6^7\}$				

20. Staircase Partitions

Staircase partitions of the form $\{N-1, N-2, \ldots, 1\}$ as characters of the unitary group $U(N)$ arise in the preceding. As representations of $U(N)$ they are of dimension

$$\prod_{i=1}^{N-1} \frac{(2(N-i))!}{(N-i)!(2N-1-2i)^i} = 2^{N(N-1)/2} \tag{20-1}$$

which is also the dimension of Δ_N in (18-3).

21. The Staircase Conjecture

If in $U(N)$ the Kronecker square

$$\Delta_N \times \Delta_N \supset c(1)_\lambda\{\lambda\} \tag{21-1}$$

then

$$\Delta_{N+1} \times \Delta_{N+1} \supset c(1)_\lambda\{2N-2, \lambda\} \tag{21-2}$$

22. $\{N-1, N-2, \ldots, 1\} \times \{N-1, N-2, \ldots, 1\}$ in U_N

Let A_N be the number of admissible partitions for $V(z_1, z_2, \ldots, z_N)^2$ as calculated by Di Francesco etal. Then it appears that the number of distinct partitions arising in the Kronecker square, $\{N-1, N-2, \ldots, 1\} \times \{N-1, N-2, \ldots, 1\}$ in U_N is A_N. These partitions are necessarily of lengths N and $N-1$. Noting the tabulation below it seems that for a given N the number of distinct partitions of length $N-1$ is A_{N-1}. We know that in the case of the square of the Vandermonde for $N \geq 8$ A_N is greater than the number of distinct partitions. In terms of the $q-$polynomials this is associated with the occurence of factors that vanish when $q = 1$. For the Kronecker square of the staircase it appears that A_N is precisely the number of admissible partitions, A_N. The Kronecker square arises in the case of $q = -1$. There are still $q-$factors involving negative coefficients of q^x but all known ones conspire to be positive. Clearly the coefficients of the irreps of U_N must be positive but could a situation arise where a coefficient for an admissible partition is zero?

N	$\ell_\lambda = N$	$\ell_\lambda = N - 1$	Total= A_N
2	1	1	2
3	3	2	5
4	11	5	16
5	43	16	59
6	188	59	247
7	864	247	1111
8	4191	1111	5302
9	21074	5302	26376

The corresponding table for the square of the Vandermonde is

N	$\ell_\lambda = N$	$\ell_\lambda = N - 1$	Total
2	1	1	2
3	3	2	5
4	11	5	16
5	43	16	59
6	188	59	247
7	864	247	1111
8	4183	1111	5294
9	21015	5294	26310
10	108971	26310	135281

Let us now consider the sums of the coefficients of the square of the Vandermonde.

N	$\ell_\lambda = N$	$\ell_\lambda = N - 1$	Total
2	-3	1	-2
3	-12	-2	-14
4	84	-14	70
5	840	70	910
6	-8190	910	-7280
7	-131040	-7280	-138320
8	1659840	-138320	1521520
9	36516480	1521520	38038000
10	-570570000	38038000	-532532000

and for the Kronecker square of staircase partition $\{N-1, N-2, \ldots, 1\}$ in U_N

N	$\ell_\lambda = N$	$\ell_\lambda = N-1$	Total
2	1	1	2
3	4	2	6
4	24	6	30
5	212	30	242
6	2942	242	3184
7	64544	3184	67728
8	2258464	67728	2326192
9	126455712	2326192	128781904

In each of the above cases one sees that the results for the partitions of length $\ell_\lambda = N-1$, for a given N case follow by adding the partition 2^{N-1} to every partition found for the $N-1$ case. This probably follows by noting that if $\ell_\lambda \geq \ell_\mu$ and if the Kronecker product $\lambda \times \mu$ is evaluated in U_N so that

$$\lambda \times \mu = \sum_\nu c^\nu_{\lambda\mu} \nu \tag{22-1}$$

then if a^N, b^N are two rectangular partitions then for the terms of length N in $U_{N+!}$ we have

$$\lambda + a^N \times \mu + b^N = \sum_\nu c^\nu_{\lambda\mu} \nu + (a+b)^N \tag{22-3}$$

For example, in U_3

$$\begin{aligned}\{53\} \times \{32\} = &\{85\} + \{841\} + \{832\} + \{76\} + 2\{751\} + 2\{742\} + \{73^2\} \\ &+ \{6^21\} + 2\{652\} + \{643\} + \{5^23\}\end{aligned}$$

while in U_4 the terms of length 3 of $\{863\} \times \{764\}$ are

$$\begin{aligned}\{863\} \times \{764\} = &\{15\ 12\ 7\} + \{15\ 11\ 8\} + \{15\ 10\ 9\} + \{14\ 13\ 7\} \\ &+ 2\{14\ 12\ 8\} + 2\{14\ 11\ 9\} + \{14\ 10^2\ \} \\ &+ \{13^2\ 8\} + 2\{13\ 12\ 9\} + \{13\ 11\ 10\ \} + \{12^2\ 10\ \}\end{aligned}$$

In that case $a = 3$ and $b = 4$.

23. A Dimension Conjecture

In $U(N)$

$$dim\{aN-a, aN-2a, aN-3a, \ldots, a\} = (a+1)^{N(N-1)/2} \tag{23-1}$$

e.g. In $U(5)$

$$dim\{24, 18, 12, 6\} = 7^{10} = 282475249$$

24. Additional note on zero coefficients

We noted on page 5 that the calculation of the coefficients of V_N^2 could be broken up into the evaluation of products of monomials $U_N^{(i)}$ with $S-$functions $V_{N-1}^2(p)$ where $i = 0, 1, 2$ with i indicating the first part of the monomials in U_N as in (11) and $p = N-2, .., 2N-4$ indicates the first part of the $S-$functions in V_{N-1}^2. Thus one can seek to evaluate the terms in

$$U_N^{(i)} V_{N-1}^2(p) \tag{24-1}$$

For what values of i, p does (24-1) contain terms whose summed total is null. For $N = 8$ one finds that none of the $S-$functions listed in Table 1 for $N = 8$ occurs for any of the permitted values of i, p while for $N = 9$ only the values of (0,10) and (2,9) yield non-zero results. Specifically,

$$\begin{array}{ll}(0,10) & 1620\{11^4\ 7^3 61\} - 72900\{11^3\ 976^3 5\} - 28800\{11^3\ 87^3 64\} \\ (2,9) & -1620\{11^4\ 7^3 61\} + 72900\{11^3\ 976^3 5\} + 28800\{11^3\ 87^3 64\}\end{array} \tag{24-2}$$

Apart from these the coefficients for all the other zeroes cancel in every case of (24-1).

25. Further comment on page 5

The results of page 5 suggest that for $N \geq 3$

$$V_N^2 = U_N V_{N-1} \tag{25-1}$$

$$= U_N^{(2)} V_{N-1} + U_N^{(1)} V_{N-1}^{(N-2)} + \sum_{x=N-2}^{N-2+[\frac{N}{3}]} U_N^{(0)} V_{N-1}^{(x)} \tag{25-2}$$

We note that the number of distinct partitions arising in the first term of (26-2) is the same as the number of distinct partitions in V_{N-1}^2. Furthermore,

$$V_{N-1}^{(N-2)} = (-1)^{[\frac{N-1}{2}]}(2N-3)!!\{(N-2)^{N-1} \tag{25-3}$$

and

$$U_N^{(1)}\{(N-2)^{N-1}\} = (-1)^{N-1} N(N-1)\{(N-1)^N\} \tag{25-4}$$

and hence the second term in (25-2) becomes

$$U_N^{(1)} V_{N-1}^{(N-2)} = (-1)^{[\frac{N}{2}]}(2N-3)!! N(N-1)\{(N-1)^N\} \tag{25-5}$$

Likewise,

$$U_N^{(0)} V_{N-1}^{(N-2)} = (-1)^{[\frac{N+2}{2}]}(2N-3)!! \frac{N(N-3)}{2}\{(N-1)^N\} \tag{25-6}$$

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