Talk to be given at the minisymposium on "Computer Algebra" of the SIMAI 98-4th National Congress of the Italian Society for Applied and Industrial Mathematics, Giardini Naxos, Sicily 1-5 June 1998

Problems in Computing Properties of Symmetric Functions and Lie Groups

Brian G. Wybourne

Instytut Fizyki, Uniwersytet Mikołaja Kopernika,
ul. Grudziądzka 5/7, 87-100 Toruń, Poland

And yet the mystery of mysteries is to view machines making machines; a spectacle that fills the mind with curious, and even awful, speculation.
— Benjamin Disraeli: *Coningsby* (1844)

What are $S-$functions?

Suppose $(\mu) = (\mu_1, \mu_2, \ldots, \mu_m)$ is a partition of an integer into integer parts μ_i then we can associate with it a *monomial*

$$\mathbf{x}^\mu = x_1^{\mu_1} x_2^{\mu_2} \ldots x_m^{\mu_m} \tag{1}$$

Example:- $\mathbf{x}^{30126} = x_1^3 x_2^0 x_3^1 x_4^2 x_5^6 \equiv x_1^3 x_3^1 x_4^2 x_5^6$

Consider a *tableau* T of shape λ then define

$$\mathbf{x}^T = \prod_{(i,j)\in\lambda} x_{T_{i,j}} = \mathbf{x}^\mu \tag{2}$$

Example:- if

$$T = \begin{array}{llll} 3 & 3 & 1 & 2 \\ 5 & 1 & 1 & \\ 2 & & & \end{array}$$

then

$$\mathbf{x}^T = x_1^3 x_2^2 x_3^2 x_5$$

A tableau T of shape λ is *semi-standard* if the integers appearing in rows are weakly increasing and strongly increasing down columns. The Schur-function ($S-$function) is defined by

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T \tag{3}$$

Example:- associated with $s_{21}(x_1, x_2, x_3)$ are the eight tableaux

$$\begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \quad \begin{array}{cc} 1 & 1 \\ 3 & \end{array}, \quad \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \quad \begin{array}{cc} 1 & 3 \\ 3 & \end{array}, \quad \begin{array}{cc} 2 & 3 \\ 3 & \end{array}, \quad \begin{array}{cc} 1 & 2 \\ 2 & \end{array}$$

$$\begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \quad \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$$

and hence

$$\begin{aligned} s_\lambda(\mathbf{x}) = {} & x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_3^2 \\ & + x_2 x_3^2 + x_1 x_2^2 + 2x_1 x_2 x_3 \end{aligned} \tag{4}$$

Placing no limits on the number of variables $\mathbf{x}$ we can write

$$s_{21}(\mathbf{x}) = \sum_{i,j} x_i x_j + 2 \sum_{i,j,r} x_i x_j x_r \tag{5}$$

where the summations are carried out over all distinct permutations of the indices. Frequently we will designate an $S-$function $s_\lambda(\mathbf{x})$ by enclosing the partition (λ) in curly brackets $\{\lambda\}$ and leave the number of variables unspecified.

$S-$functions are *symmetric functions*, thus their products and powers may be resolved into sums of $S-$functions.

What can you do with $S-$functions?

1. **Outer Products**

$$\{\mu\} \cdot \{\nu\} = \sum_{\lambda} c^{\lambda}_{\mu\nu}\{\lambda\} \tag{1}$$

where the $c^{\lambda}_{\mu\nu}$ are non-negative integers known as the Littlewood-Richardson coefficients and the weights, ω_λ, are constrained by $\omega_\lambda = \omega_\mu + \omega_\nu$

2. **Skews**

$$\{\lambda/\mu\} = \sum_{\nu} c^{\lambda}_{\mu\nu}\{\lambda\} \tag{2}$$

The weights, ω_ν, are constrained by $\omega_\nu = \omega_\lambda - \omega_\mu$

3. **Plethysms**

$$\{\lambda\} \otimes \{\mu\} = \sum_{\nu} g^{\nu}_{\lambda\mu}\{\nu\} \tag{3}$$

where the $g^{\nu}_{\lambda\mu}$ are non-negative integers and the weights, ω_ν, are constrained by $\omega_\nu = \omega_\lambda \times \omega_\mu$

4. **Inner Products**

$$\{\mu\} * \{\nu\} = \sum_{\lambda} c^{\lambda}_{\mu\nu} \{\lambda\} \tag{4}$$

where the $c^{\lambda}_{\mu\nu}$ are non-negative integers and the weights of the partitions are constrained by $\omega_\mu = \omega_\nu = \omega_\lambda = n$.

5. **Inner Plethysms**

$$\{\mu\} \circ \{\nu\} = \sum_{\lambda} c^{\lambda}_{\mu\nu} \{\lambda\} \tag{5}$$

where the $c^{\lambda}_{\mu\nu}$ are non-negative integers and the weights of the partitions are constrained by $\omega_\mu = \omega_\lambda = n$ and $\omega_\nu \geq 0$.

Examples with SCHUR

```
SFN>
o21,32
    {53} + {521} + {4^2 } + 2{431}
  + {42^2 } + {421^2 } + {3^2 2} + {3^2 1^2 }
  + {32^2 1}
SFN>
sk321,21
      {3} + 2{21} + {1^3 }
SFN>
 pl21,3
      {63} + {531} + {52^2 } + {521^2 }
    + {4^2 1} + {432} + {431^2 } + 2{42^2 1}
    + {421^3 } + {41^5 } + {3^3 } + {3^2 21}
    + {3^2 1^3 } + {32^3 } + {32^2 1^2 }
SFN>
i32,2111
      {32} + {31^2 } + {2^2 1} + {21^3 }
SFN>
i_pl21
      <21> + <2> + <1^2 > + <1>
SFN>
```

Some infinite series of $S-$functions

$$L = \sum_{m=0}^{\infty} (-1)^m \{1^m\} \quad M = \sum_{m=0}^{\infty} \{m\}$$
$$P = \sum_{m=0}^{\infty} (-1)^m \{m\} \quad Q = \sum_{m=0}^{\infty} \{1^m\}$$
$$B = \sum_{\beta} \{\beta\} \quad D = \sum_{\delta} \{\delta\}$$

where the m are integers, the partitions (δ) are all partitions having only even parts while the partitions (β) are conjugates of the (δ). L and M are inverses of one another as are P and Q.

SCHUR can compute these series, and many others, up to a user determined limit.

Examples:-

```
SFN>
ser6,b
      {3^2 } + {2^2 1^2 } + {2^2 } + {1^6 }
    + {1^4 } + {1^2 } + {0}
SFN>
ser6,d
      {6} + {42} + {4} + {2^3 } + {2^2 }
    + {2} + {0}
SFN>
```

New $S-$function identities

Infinite $S-$function series play a key role in practical calculations for both compact and non-compact Lie groups. SCHUR gave evidence leading to a number of conjectures involving plethysms of certain infinite $S-$functions.

$$M_+ = \sum_{m=0}^{\infty}\{2m\} \quad M_- = \sum_{m=0}^{\infty}\{2m+1\}$$
$$L_+ = \sum_{m=0}^{\infty}\{1^{2m}\} \quad L_- = \sum_{m=0}^{\infty}\{1^{2m+1}\} \tag{1}$$

$$A_\pm = \{1^2\}\otimes L_\pm \quad B_\pm = \{1^2\}\otimes M_\pm$$
$$C_\pm = \{2\}\otimes L_\pm \quad D_\pm = \{2\}\otimes M_\pm \tag{2}$$

Let $Z_\pm = \{A_\pm, B_\pm, C_\pm, D_\pm\}$ then

$$Z_+\otimes\{1^2\} = Z_-\otimes\{2\}$$
$$Z_+\otimes\{21^2\} = Z_-\otimes\{31\} \tag{3}$$

1. M Yang and B G Wybourne, J. Phys. A: Math. Gen. **19** 3513 (1986)
2. R C King, B G Wybourne and M Yang, J. Phys. A: Math. Gen. **22** 4519 (1989)
3. K Grudzinski and B G Wybourne, J. Phys. A: Math. Gen. **29** 6631 (1996)

A tensor product in $SO(10)$

$$[\lambda]\times[\mu]=\sum_{\zeta}[\lambda/\zeta\cdot\mu/\zeta] \tag{1}$$

Example:-

$$[1^3]\times[2^3]=\sum_{\zeta}[1^3/\zeta\cdot 2^3/\zeta]$$
$$[1^3\cdot 1^3]+[1^2\cdot 2^2 1]+[1\cdot 21^2]+[0\cdot 1^3]$$

$$\begin{aligned}
[1^3\cdot 2^3]&=[3^3]+[3^2 21]+[32^2 1^2]+[2^3 1^3]\\
[1^2\cdot 2^2 1]&=[3^2 1]+[32^2]+[321^2]+[2^3 1]+[2^2 1^3]\\
[1\cdot 21^2]&=[31^2]+[2^2 1]+[21^3]\\
[0\cdot 1^3]&=[1^3]
\end{aligned}$$

$$\begin{aligned}
[32^2 1^2]\equiv&[32^2 1^2]_+ + [32^2 1^2]_-\\
[2^3 1^3]\equiv&[2^3 1]\\
[2^2 1^3]\equiv&[2^2 1^3]_+ + [2^2 1^3]_-
\end{aligned}$$

$$\begin{aligned}
[1^3]\times[2^3]=&[3^3]+[3^2 21]+[3^2 1]+[32^2 1^2]+\\
&+[32^2 1^2]-+[32^2]+[321^2]+[31^2]\\
&+2[2^3 1]+[2^2 1^3]++[2^2 1^3]-+[2^2 1]\\
&+[21^3]+[1^3]
\end{aligned}$$

Calculation by SCHUR

```
gr so10
Group is SO(10)
REP>
p111,222
  [3^3 ] + [3^2 21] + [3^2 1] + [32^2 1^2 ]+
+ [32^2 1^2 ]- + [32^2 ] + [321^2 ] + [31^2 ]
+ 2[2^3 1] + [2^2 1^3 ]+ + [2^2 1^3 ]-
+ [2^2 1] + [21^3 ] + [1^3 ]
REP>
dim last
dimension=495000
```

1. Time taken to compute the result by hand < 2 minutes.
2. Time taken by SCHUR on a Pentium instantaneous.
3. Time reported in the literature on a VAX4000 5 hours CPU.

The product $[6^3] \times [9^3]$ is of dimension 92,908,920,088,670,400. SCHUR resolved the product in 40 minutes on a SUN IPX. Undoubtedly the Vax4000 programme would take a time that would dwarf the age of the universe!

$S-$function series and branching rules

$$U(n)\downarrow U(n-1)$$
$$\{\lambda\}\downarrow\{\lambda/M\}$$

$$U(n)\downarrow O(n)$$
$$\{\lambda\}\downarrow[\lambda/D]$$

$$U(2n)\downarrow Sp(2n)$$
$$\{\lambda\}\downarrow\langle\lambda/B\rangle$$

$$Sp(2n,R)\downarrow U(n)$$
$$\langle\frac{k}{2}(\lambda)\rangle\downarrow\varepsilon^{\frac{k}{2}}\cdot\{\{\lambda_s\}_N^k\cdot D_N\}_N \quad N=\min(k,n)$$

$$SO^*(2n)\downarrow U(n)$$
$$[k(\lambda)]\downarrow\varepsilon^k\cdot\{\{\lambda_s\}_N^{2k}\cdot B_N\}_N \quad N=\min(2k,n)$$

Examples of branching rules with SCHUR

```
DP>
gr u4
Group is U(4)
DP>
br1,4gr1[321]
Group is O(4)
      [31] + [2^2 ] + [2]# + [2] + [1^2 ]
DP>
gr u4
Group is U(4)
DP>
br2,4gr1[321]
Group is Sp(4)
      <31> + <2^2 > + <2> + <1^2 >
DP>
gr spr6
Group is Sp(6,R)
DP>
br36,6gr1[2;21]
Group is U(3)
 {432} + {4^2 3} + {53^2 } + {542} + {54^2 }
 + {5^2 3} + {632} + 2{643} + {652} + 2{654}
 + {6^2 3} + {6^2 5} + {73^2 } + {742}
 + {74^2 } + 2{753} + {75^2 } + {762} +...
```

$S-$functions and tensor products

$$U(n) : \{\mu\} \times \{\nu\} = \sum_{\lambda} C^{\lambda}_{\mu\nu}\{\lambda\}$$

$$O(n) : [\mu] \times [\nu] = \sum_{\zeta} [\mu/\zeta \cdot \nu/\zeta]$$

$$Sp(2n) : \langle\mu\rangle \times \langle\nu\rangle = \sum_{\zeta} \langle\mu/\zeta \cdot \nu/\zeta\rangle$$

$$Sp(2n, R) : \langle\frac{k}{2}(\mu)\rangle \times \langle\frac{\ell}{2}(\nu)\rangle = \langle\frac{k+\ell}{2}(\{\mu_s\}^k \cdot \{\nu_s\}^\ell \cdot D)_{k+\ell,n}\rangle$$

$$SO^*(2n) : [k(\mu)] \times [\ell(\nu)] = [k+\ell(\{\mu_s\}^{2k} \cdot \{\nu_s\}^{2\ell} \cdot B)_{k+\ell,n}]$$

Examples of tensor products with SCHUR

```
REP>
gr sp6
Group is Sp(6)
REP>
p21,31
  <52> + <51^2 > + <5> + <43> + 2<421>
 + 3<41> + <3^2 1> + <32^2 > + 3<32>
 + 3<31^2 > + 2<3> + 2<2^2 1> + 3<21>
 + <1^3 > + <1>
REP>
gr so8
Group is SO(8)
REP>
p s;0+,21
 [s;21]+ + [s;2]- + [s;1^2 ]- + [s;1]+
REP>
gr spr6
Group is Sp(6,R)
REP>
p1;0,2;1
  <3;(1)> + <3;(1^3 )> + 2<3;(21)>
 + <3;(2^2 1)> + <3;(3)> + <3;(31^2 )>
 + 2<3;(32)> + <3;(3^2 1)> + 2<3;(41)>
 + <3;(421)> + 2<3;(43)> + ...
```

Algebraic approaches to the genetic code

Hornos and Hornos[1] investigated those simple Lie algebras having at least one representation of dimension 64, the number 64 corresponding to the $4 \times 4 \times 4$ possible codons, each involving four bases arranged in triplets, to code the 20 amino acids.

The groups $Sp(6)$ and $G(2)$ were found[1,2] to be of particular interest. SCHUR has been able to determine the various possible group-subgroup decompositions and the eigenvalues of the Casimir operators used to describe the possible symmetry breakings.

In addition SCHUR was used to establish the complete set of 64−dimensional representations for the symmetric and alternating groups.

The complete set of 64−dimensional irreducible representations for the groups $S(n)$ and $A(n)$

$S(8)$	$\{521\}$	$\{321^3\}$	$S(13)$	$\{\Delta\}$
$A(8)$	$[521]$		$S(14)$	$\{\Delta_\pm\}$
$S(65)$	$\{64\ 1\}$	$\{21^{63}\}$	$A(14)$	$[64\ 1]$
$A(65)$	$[64\ 1]$		$A(15)$	$[\Delta_\pm]$

1. J E M Hornos and Y M M Hornos, Phys. Rev. Lett. **71** 4401 (1993)
2. M Forger, Y M M Hornos and J E M Hornos, Phys. Rev. **E56** 7078 (1997)
3. R D Kent, M Schlesinger and B G Wybourne, Can. J. Phys. (In Press)

Generating functions for stable branching coefficients of $U(n) \downarrow S_n, O(n) \downarrow S_n$ and $O(n-1) \downarrow S_n$

Problems in symplectic models of nuclei, quantum dots and many-electron states often involve the symmteric group S_n. Applications require the resolution of symmetrised powers of tensor representations of S_n. These are required in determining branching coefficients. The coefficients involve *inner plethysms*. Of particular interest is the representation

$$\{n-1,1\} \equiv \langle 1 \rangle \tag{1}$$

and the inner plethysms

$$\langle 1 \rangle \otimes \{\lambda\} = \sum_{\rho} c_{\lambda}^{\rho} \langle \rho \rangle \tag{2}$$

SCHUR has computed the complete resolution of the plethysms $\langle 1 \rangle \otimes \{n\}$ for $n = 1, \ldots, 20$. Applications often require the value of single coefficients in very large plethysms. Here generating methods can be used. Thus with MAPLE it was possible to show that in S_{40}

$$\{39,1\} \otimes \{30\ 4321\} \supset 309,727,790,880\{31\ 3^2 2\}$$

A calculation quite beyond SCHUR.

1. T Scharf, J-Y Thibon and B G Wybourne, J. Phys. A: Math. Gen. **26** 7461 (1993)
2. T Scharf, J-Y Thibon and B G Wybourne, J. Phys. A: Math. Gen. **30** 6963 (1997)

The Vandermonde determinant and the quantum Hall effect

The Vandermonde alternating function in N variables is defined as

$$V(z_1,\ldots,z_N)=\prod_{i<j}^{N}(z_i-z_j) \tag{1}$$

Any even power, V^{2m}, is necessarily a *symmetric* function and hence expandable into a set of symmetric functions such as the Schur functions

$$s_\lambda(z_1,\ldots,z_N)=\{\lambda\}=\{\lambda_1,\ldots,\lambda_p\} \tag{2}$$

which in this case are indexed by partitions of the integer

$$n=mN(N-1) \tag{3}$$

We need the expansion coefficients c^λ for

$$V^{2m}=\sum_{\lambda\vdash n}c^\lambda s_\lambda \tag{4}$$

where the c^λ are signed integers and are precisely the same integers that arise in the expansion of the Laughlin wavefunction, used in the quantum Hall effect, as a linear combination of Slater determinants.

This is a COMBINATORIALLY EXPLOSIVE problem!

1. T Scharf, J-Y Thibon and B G Wybourne, J. Phys. A: Math. Gen. **27** 4211 (1994).

N	$N_{tableaux}$	$N_{tableaux}^{conjectured*}$	N_{coeff}
1	1	1	1
2	2	2	4
3	5	5	28
4	16	16	292
5	59	59	4,102
6	247	247	73,444
7	1,111	1,111	1,605,838
8	5,294	5,302	41,603,200
9	26,310	26,376	

* *"The above reasoning does not however insure that this is exactly the total number of tableaux in the expansion of V^{2m} in characters as some coefficients might still vanish. However experience up to $N = 5$ seems to indicate that these accidents do not happen"* P. Di Francesco, M. Gaudin, C. Itzykson and F. Lesage. (SphT/93-125)

Invariants formed from the Riemann tensor

The master object for enumerating Riemann scalars is

$$\mathcal{G} = \sum_{m=1}^{\infty} (t^2\{2^2\} + t^3\{32\} + t^4\{42\} + \ldots + t^p\{p,2\} + \ldots)^m \quad (1)$$

1. There is a Riemann scalar for every $S-$function $\{\lambda\}$ arising in (1) whose partition label $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_p$ involves only *even* parts.
2. The evaluation of the Riemann scalars of order n involves the resolution of all plethysms and outer $S-$function products associated with t^n where n is necessarily even.

Order n	Number of Riemann Scalars
2	1
4	4
6	17
8	92
10	668
12	6,721
14	89,137

1. S A Fulling, R C King, B G Wybourne and C J Cummins, Class. Quantum Grav. **9** 1151 (1992)
2. B G Wybourne and J Meller, J. Phys. A: Math. Gen. **25** 5999 (1992)

Collaborators

Prof R C King, Mathematics Department, Southampton University, UK

Prof J-Y Thibon and F Toumazet, Institut Gaspard Monge, Université de Marne-la-Vallée, France

Dr T Scharf, Lehrstuhl II für Mathematik, Universtät Bayreuth, Germany

Prof M Schlesinger, Department of Physics, University of Windsor, Canada

---

Research supported by Polish KBN Grants

Questions?

The only questions worth asking are the unanswerable ones
— John Ciardi *Saturday Review-World (1973)*

For every complex question there is a simple answer — and it's wrong.
— H. L. Mencken