Calculating properties of the non-compact group $U(p,q)$ with SCHUR

1. Introduction

The problem of computing Kronecker products and branching rules for the non-compact group $U(p,q)$ was considered some time ago by King and Wybourne[1] (referred herein as KW) and more recently by Thibon etal[2] (referred herein as TTW). That work is not, as yet, built into SCHUR but nevertheless it is possible to use SCHUR to calculate various properties of $U(p,q)$. Here we outline how to compute Kronecker products for $U(p,q)$

2. The harmonic series unirreps of $U(p,q)$

KW show that harmonic series unirreps of $U(p,q)$ can be generated from considering the powers H^k of the fundamental unirreps H leading to the labelling of a typical harmonic series unirreps as

$$\{k(\bar{\nu};\mu)\} \tag{1}$$

where the partitions (ν) and (μ) have, respectively, at most p and q parts with the added constraints that the conjugate partitions $(\tilde{\nu})$ and $(\tilde{\mu})$ satisfy

$$\tilde{\nu}_1 + \tilde{\mu}_1 \leq k \tag{2a}$$

and

$$\tilde{\nu}_1 \leq p \quad \text{and} \quad \tilde{\mu}_1 \leq q \tag{2b}$$

3. The basic Kronecker product result

Using such a notation, KW (9.7) resolves the Kronecker product of two arbitrary unirreps as

$$\{k(\bar{\nu};\mu)\} \times \{\ell(\bar{\tau};\sigma)\} = \sum_{\zeta} \{k+\ell((\{\bar{\nu}_s\}^k \cdot \{\bar{\tau}_s\}^\ell \cdot \{\bar{\zeta}\}; \{\mu_s\}^k \cdot \{\sigma_s\}^\ell \cdot \{\zeta\}))\} \tag{3}$$

where it is understood that terms are paired together so that $\{\bar{\nu}_s;\mu_s\}^k$ and $\{\bar{\tau}_s;\sigma_s\}^\ell$ are *signed sequences*[3] calculated through the use of the modification rules of $U(k)$ and $U(\ell)$ respectively.

As in Eq. (2) it is understood that

$$((\bar{\rho};\lambda))_{k+\ell,p,q} = \begin{cases} (\bar{\rho};\lambda) & \text{if } \tilde{\lambda}_1 \leq p, \tilde{\rho}_1 \leq q \quad \text{and} \quad \tilde{\lambda}_1 + \tilde{\rho}_1 \leq k+\ell \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We now illustrate the various steps that must be taken to compute the Kronecker product:-

$$\{2(\bar{3};1)\} \times \{2(\bar{1};2)\} \tag{5}$$

for the groups $U(2,2)$ and $U(4,4)$

4. Determination of the signed sequences

The first step in implementing Eq. (3) is to determine the required terms in the signed sequences. We first note that an irrep $\{\bar{\nu};\mu\}$ is standard in $U(k)$ if, and only if,

$$\ell_\nu + \ell_\mu \leq k \tag{6}$$

all other irreps are either null or reducible to a signed standard irrep. A non-standard irrep of $U(k)$ may be modified by apply the modification rule[4]

$$\{\bar{\nu};\mu\} = (-1)^{c+d-1}\{\overline{\nu-h};\mu-h\} \quad h = \ell_\nu + \ell_\mu - k - 1 \geq 0 \tag{7}$$

In applying the rule one removes from the partition (ν) h boxes as a continuous strip starting at the foot of the first column and ending in the c-th column and similarly for the partition (μ) ending with the d-th column. This process is continued until either ordered partitions or a null result is obtained. This is done automatically in SCHUR using the command "std". Thus for example we have the signed sequence for $\{\bar{3};1\}^2_s$ and $\{\bar{1};2\}^2_s$ as

$$\begin{array}{llllll} \{\bar{3};1\}^2_s = & (\bar{3};1) & -\ (\overline{31};1^2) & +\ (\overline{32};1^3) & +\ (\overline{31^3};2^2) & -\ (\overline{321^2};2^21) \\ & -\ (\overline{31^4};32) & +\ (\overline{31^5};42) & +\ (\overline{321^3};321) & -\ (\overline{3^21^3};321^2) & -\ (\overline{31^6};52) \\ & -\ (\overline{321^4};421) & +\ (\overline{31^7};62) & +\ (\overline{321^5};521) & +\ (\overline{3^21^4};421^2) & -\ (\overline{31^8};72) \\ & -\ (\overline{321^6};621) & -\ (\overline{3^21^5};521^2) & +\ (\overline{4^21^3};321^4) & -\ (\overline{74};1^9) & \ldots \end{array} \tag{8a}$$

$$
\begin{array}{llllll}
\{\bar{1};2\}^2_s = & (\bar{1};2) & -\,(\overline{1^2};21) & +\,(\overline{1^3};2^2) & +\,(\overline{2^2};21^3) & -\,(\overline{2^21};2^21^2) \\
 & -\,(\overline{1^5};3^2) & +\,(\overline{321};2^21^3) & +\,(\overline{1^6};43) & +\,(\overline{2^21^3};3^21^2) & -\,(\overline{1^7};53) \\
 & -\,(\overline{421};2^21^4) & +\,(\overline{1^8};63) & -\,(\overline{2^21^4};431^2) & +\,(\overline{521};2^21^5) & -\,(\overline{1^9};73) \\
 & +\,(\overline{2^21^5};531^2) & -\,(\overline{621};2^21^6) & +\,(\overline{421^3};3^21^4) & -\,(\overline{7}221^8) & \ldots
\end{array}
\qquad (8b)
$$

The above signed sequences can be verified in SCHUR as in the following SCHUR fragment

```
DPrep Mode (with function)
DP>
rep
REP mode
REP>
gr u2
Group is U(2)
REP>
std31;11
       - {3;1}
REP>
std32;111
      {3;1}
REP>
std11;21
       - {1;2}
REP>
std111;22
      {1;2}
REP>
```

Note that the signed sequences are infinite sequences but in practice the number of terms is rendered finite for finite values of p and q as well as those of k and ℓ.

5. The product of the signed sequences

The next step is to form the product of the two signed sequences for the group $U(p)\times U(q)$ while at the same time satisfying the constraints imposed by Eq. (4). In our case this restricts each of the signed sequences to just the first two terms. Thus for $U(4,4)$ we have the SCHUR fragment:-

```
DP>
gr2u4u4
Groups are   U(4) * U(4)
DP>
p[3*1]-[31*11],[1*2]-[11*21]
 {42}{32} + {42}{31^2 } + {42}{2^2 1} + {42}{21^3 }
       + {41^2 }{32} + {41^2 }{31^2 } + {41^2 }{2^2 1}
       + {41^2 }{21^3 } - 2{41}{31} - {41}{2^2 }
       - 2{41}{21^2 } + {4}{3} + {4}{21} + {321}{32}
       + {321}{31^2 } + {321}{2^2 1} + {321}{21^3 }
       - {32}{31} - {32}{21^2 } + {31^3 }{32}
       + {31^3 }{31^2 } + {31^3 }{2^2 1} + {31^3 }{21^3 }
```

```
     - 2{31^2 }{31} - {31^2 }{2^2 } - 2{31^2 }{21^2 }
     + {31}{3} + {31}{21}
   DP>
```

The constraints of Eq. (4) effectively eliminate all $U(4)$ irreps involving partitions with 3 non-zero parts. These are eliminated in the next SCHUR fragment:-

```
DP>
len1,2last
      {42}{32} + {42}{31^2 } + {42}{2^2 1} + {42}{21^3 }
       - 2{41}{31} - {41}{2^2 } - 2{41}{21^2 } + {4}{3}
       + {4}{21} - {32}{31} - {32}{21^2 } + {31}{3}
       + {31}{21}
DP>
len2,2last
      {42}{32} - 2{41}{31} - {41}{2^2 } + {4}{3}
       + {4}{21} - {32}{31} + {31}{3} + {31}{21}
DP>
setp1last
DP>
```

These are all the terms required for the product of the two signed sequences. Note we have saved the terms as v1 for later use.

6. The sum $\sum_\zeta \{\bar{\zeta}\} \times \{\zeta\}$

The next step is to prepare the terms for the sum $\sum_\zeta \{\bar{\zeta}\} \times \{\zeta\}$. First note that Eq. (4) restricts the partitions (ζ) to at most 2 non-zero parts. The series is infinite and a suitable cutoff must be chosen. We shall restrict ourselves to terms (ζ) of weight ≤ 6. The relevant (ζ) may be determined by the following SCHUR fragment:-

```
SFN>
len2wt6ser6,f
      {6} + {51} + {5} + {42} + {41} + {4} + {3^2 } + {32}
       + {31} + {3} + {2^2 } + {21} + {2} + {1^2 } + {1}
       + {0}
SFN>
```

We now return to the DPmode and prepare v2 as the appropriate list as shown in the SCHUR fragment:-

```
exit
DPrep Mode (with function)
Groups are   U(4) * U(4)
DP>
setp2[0*0]+[1*1]+[2*2]+[11*11]+[3*3]+[21*21]+[4*4]+[31*31]+[22*22]+[5*5]
DP>
setp2add v2,[41*41]+[32*32]+[6*6]+[51*51]+[42*42]+[33*33]
DP>
v2
      {6}{6} + {51}{51} + {5}{5} + {42}{42} + {41}{41}
       + {4}{4} + {3^2 }{3^2 } + {32}{32} + {31}{31}
       + {3}{3} + {2^2 }{2^2 } + {21}{21} + {2}{2}
```

```
        + {1^2 }{1^2 } + {1}{1} + {0}{0}
DP>
```

7. The product obtained

We now form the product of v1 with v2 in $U(4) \times U(4)$ keeping terms of weight up to 6 via the following SCHUR fragment:-

```
set_pwt6
DP>
wt1,6p v1,v2
 {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
        + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
        + 2{51}{31^2 } + 2{51}{2^2 1} + {51}{21^3 } + {5}{4}
        + 2{5}{31} + {5}{2^2 } + {5}{21^2 } + 2{42}{5}
        + 2{42}{41} + 2{42}{32} + {42}{31^2 } + 2{42}{2^2 1}
        + {42}{21^3 } + {41^2 }{5} + 2{41^2 }{41}
        + {41^2 }{32} + 3{41^2 }{31^2 } + 2{41^2 }{2^2 1}
        + 2{41^2 }{21^3 } + 2{41}{4} + 2{41}{31}
        + {41}{2^2 } + 2{41}{21^2 } + {4}{3} + {4}{21}
        + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
        + {3^2 }{2^2 1} + {321}{5} + 2{321}{41} + 2{321}{32}
        + 2{321}{31^2 } + 2{321}{2^2 1} + {321}{21^3 }
        + {32}{4} + {32}{31} + {32}{2^2 } + {32}{21^2 }
        + {31^3 }{41} + {31^3 }{32} + 2{31^3 }{31^2 }
        + {31^3 }{2^2 1} + {31^3 }{21^3 } + {31^2 }{4}
        + 2{31^2 }{31} + {31^2 }{2^2 } + {31^2 }{21^2 }
        + {31}{3} + {31}{21}
DP>
setp3last
DP>
```

where we have saved the result as v3. Again we must consider Eq. (4) which allows us to eliminate all cases where both partitions have 3 non-zero parts, those where one has 3 non-zero parts and the other 2 non-zero parts and those where one partition has 4 non-zero parts with the other also having non-zero parts. These latter are eliminated in the following SCHUR fragment:-

```
DP>
sub v3,len1,-4v3
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
        + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
        + 2{51}{31^2 } + 2{51}{2^2 1} + {51}{21^3 } + {5}{4}
        + 2{5}{31} + {5}{2^2 } + {5}{21^2 } + 2{42}{5}
        + 2{42}{41} + 2{42}{32} + {42}{31^2 } + 2{42}{2^2 1}
        + {42}{21^3 } + {41^2 }{5} + 2{41^2 }{41}
        + {41^2 }{32} + 3{41^2 }{31^2 } + 2{41^2 }{2^2 1}
        + 2{41^2 }{21^3 } + 2{41}{4} + 2{41}{31}
        + {41}{2^2 } + 2{41}{21^2 } + {4}{3} + {4}{21}
        + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
```

```
        + {3^2 }{2^2 1} + {321}{5} + 2{321}{41} + 2{321}{32}
        + 2{321}{31^2 } + 2{321}{2^2 1} + {321}{21^3 }
        + {32}{4} + {32}{31} + {32}{2^2 } + {32}{21^2 }
        + {31^2 }{4} + 2{31^2 }{31} + {31^2 }{2^2 }
        + {31^2 }{21^2 } + {31}{3} + {31}{21}
DP>
sub last,len2,-4last
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
       + 2{51}{31^2 } + 2{51}{2^2 1} + {5}{4} + 2{5}{31}
       + {5}{2^2 } + {5}{21^2 } + 2{42}{5} + 2{42}{41}
       + 2{42}{32} + {42}{31^2 } + 2{42}{2^2 1}
       + {41^2 }{5} + 2{41^2 }{41} + {41^2 }{32}
       + 3{41^2 }{31^2 } + 2{41^2 }{2^2 1} + 2{41}{4}
       + 2{41}{31} + {41}{2^2 } + 2{41}{21^2 } + {4}{3}
       + {4}{21} + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {3^2 }{2^2 1} + {321}{5} + 2{321}{41} + 2{321}{32}
       + 2{321}{31^2 } + 2{321}{2^2 1} + {32}{4} + {32}{31}
       + {32}{2^2 } + {32}{21^2 } + {31^2 }{4}
       + 2{31^2 }{31} + {31^2 }{2^2 } + {31^2 }{21^2 }
       + {31}{3} + {31}{21}
DP>
setp3last
DP>
```

The next step is to keep the cases where one partition has 3 non-zero parts and the other 1 nonzero part. This may be done by first putting all these cases into a single variable v4 as indicated by the following SCHUR fragment:-

```
DP>
len1,1,v3
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + {5}{4} + 2{5}{31} + {5}{2^2 }
       + {5}{21^2 } + {4}{3} + {4}{21}
DP>
add last,len2,1v3
      2{6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 2{5}{4} + 2{5}{31}
       + {5}{2^2 } + {5}{21^2 } + 2{42}{5} + {41^2 }{5}
       + 2{41}{4} + 2{4}{3} + {4}{21} + {3^2 }{5}
       + {321}{5} + {32}{4} + {31^2 }{4} + {31}{3}
DP>
add len1,-3last,len2,-3last
      {6}{31^2 } + {6}{2^2 1} + {5}{21^2 } + {41^2 }{5}
       + {321}{5} + {31^2 }{4}
DP>
```

```
setp4last
DP>
```

Now remove from v3 all cases involving at least one partition into 3 non-zero parts and then adding back the cases saved as v4 as shown below:-

```
DP>
len1,2v3
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
       + 2{51}{31^2 } + 2{51}{2^2 1} + {5}{4} + 2{5}{31}
       + {5}{2^2 } + {5}{21^2 } + 2{42}{5} + 2{42}{41}
       + 2{42}{32} + {42}{31^2 } + 2{42}{2^2 1} + 2{41}{4}
       + 2{41}{31} + {41}{2^2 } + 2{41}{21^2 } + {4}{3}
       + {4}{21} + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {3^2 }{2^2 1} + {32}{4} + {32}{31} + {32}{2^2 }
       + {32}{21^2 } + {31}{3} + {31}{21}
DP>
len2,2last
      {6}{5} + 2{6}{41} + 2{6}{32} + 2{51}{5}
       + 3{51}{41} + 2{51}{32} + {5}{4} + 2{5}{31}
       + {5}{2^2 } + 2{42}{5} + 2{42}{41} + 2{42}{32}
       + 2{41}{4} + 2{41}{31} + {41}{2^2 } + {4}{3}
       + {4}{21} + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {32}{4} + {32}{31} + {32}{2^2 } + {31}{3}
       + {31}{21}
DP>
add last,v4
      {6}{5} + 2{6}{41} + 2{6}{32} + {6}{31^2 }
       + {6}{2^2 1} + 2{51}{5} + 3{51}{41} + 2{51}{32}
       + {5}{4} + 2{5}{31} + {5}{2^2 } + {5}{21^2 }
       + 2{42}{5} + 2{42}{41} + 2{42}{32} + {41^2 }{5}
       + 2{41}{4} + 2{41}{31} + {41}{2^2 } + {4}{3}
       + {4}{21} + {3^2 }{5} + {3^2 }{41} + {3^2 }{32}
       + {321}{5} + {32}{4} + {32}{31} + {32}{2^2 }
DP>
setp4last
DP>
```

which is the desired result saved as v4. In the case of $U(2,2)$ we simply eliminate all terms involving 3 non-zero parts and store the result as v5.

```
DP>
sb_rev true
DP>
columns5
DP>
len1,2last
```

```
      {31}{21} + {31}{3} + {32}{2^2 } + {32}{31}
       + {32}{4} + {3^2 }{32} + {3^2 }{41} + {3^2 }{5}
       + {4}{21} + {4}{3} + {41}{2^2 } + 2{41}{31}
       + 2{41}{4} + 2{42}{32} + 2{42}{41} + 2{42}{5}
       + {5}{21^2 } + {5}{2^2 } + 2{5}{31} + {5}{4}
       + 2{51}{32} + 3{51}{41} + 2{51}{5} + {6}{2^2 1}
       + {6}{31^2 } + 2{6}{32} + 2{6}{41} + {6}{5}
DP>
len2,2last
      {31}{21} + {31}{3} + {32}{2^2 } + {32}{31}
       + {32}{4} + {3^2 }{32} + {3^2 }{41} + {3^2 }{5}
       + {4}{21} + {4}{3} + {41}{2^2 } + 2{41}{31}
       + 2{41}{4} + 2{42}{32} + 2{42}{41} + 2{42}{5}
       + {5}{2^2 } + 2{5}{31} + {5}{4} + 2{51}{32}
       + 3{51}{41} + 2{51}{5} + 2{6}{32} + 2{6}{41}
       + {6}{5}
DP>
setp5last
DP>
```

We now prepare the final result as TEX output as shown below:-

```
DP>
v4
      \+$\{31\}\{21\}$&$ + \ \{31\}\{3\}$&$ + \ \{31^2\}\{4\}$&
$ + \ \{32\}\{2^2\}$&$ + \ \{32\}\{31\}$\cr
      \+$ + \ \{32\}\{4\}$&$ + \ \{321\}\{5\}$&$ + \ \{3^2\}\{32\}$&
$ + \ \{3^2\}\{41\}$&$ + \ \{3^2\}\{5\}$\cr
      \+$ + \ \{4\}\{21\}$&$ + \ \{4\}\{3\}$&$ + \ \{41\}\{2^2\}$&
$ + \ 2\{41\}\{31\}$&$ + \ 2\{41\}\{4\}$\cr
      \+$ + \ \{41^2\}\{5\}$&$ + \ 2\{42\}\{32\}$&$ + \ 2\{42\}\{41\}$&
$ + \ 2\{42\}\{5\}$&$ + \ \{5\}\{21^2\}$\cr
      \+$ + \ \{5\}\{2^2\}$&$ + \ 2\{5\}\{31\}$&$ + \ \{5\}\{4\}$&
$ + \ 2\{51\}\{32\}$&$ + \ 3\{51\}\{41\}$\cr
      \+$ + \ 2\{51\}\{5\}$&$ + \ \{6\}\{2^21\}$&$ + \ \{6\}\{31^2\}$&
$ + \ 2\{6\}\{32\}$&$ + \ 2\{6\}\{41\}$\cr
      \+$ + \ \{6\}\{5\}$\cr
DP>
v5
      \+$\{31\}\{21\}$&$ + \ \{31\}\{3\}$&$ + \ \{32\}\{2^2\}$&
$ + \ \{32\}\{31\}$&$ + \ \{32\}\{4\}$\cr
      \+$ + \ \{3^2\}\{32\}$&$ + \ \{3^2\}\{41\}$&$ + \ \{3^2\}\{5\}$&
$ + \ \{4\}\{21\}$&$ + \ \{4\}\{3\}$\cr
      \+$ + \ \{41\}\{2^2\}$&$ + \ 2\{41\}\{31\}$&$ + \ 2\{41\}\{4\}$&
$ + \ 2\{42\}\{32\}$&$ + \ 2\{42\}\{41\}$\cr
```

```
      \+$ + \ 2\{42\}\{5\}$&$ + \ \{5\}\{2^2\}$&$ + \ 2\{5\}\{31\}$&
$ + \ \{5\}\{4\}$&$ + \ 2\{51\}\{32\}$\cr
      \+$ + \ 3\{51\}\{41\}$&$ + \ 2\{51\}\{5\}$&$ + \ 2\{6\}\{32\}$&
$ + \ 2\{6\}\{41\}$&$ + \ \{6\}\{5\}$\cr
DP>
```

which may be text edited to produce the final results as:-

Terms to weight 6 in the product $\{2(\overline{3};1)\} \times \{2(\overline{1};2)\}$ for $U(2,2)$ and $U(4,4)$

Result for $U(4,4)$

$\{4(\overline{31};21)\}$	$+\ \{4(\overline{31};3)\}$	$+\ \{4(\overline{31^2};4)\}$	$+\ \{4(\overline{32};2^2)\}$	$+\ \{4(\overline{32};31)\}$
$+\ \{4(\overline{32};4)\}$	$+\ \{4(\overline{321};5)\}$	$+\ \{4(\overline{3^2};32)\}$	$+\ \{4(\overline{3^2};41)\}$	$+\ \{4(\overline{3^2};5)\}$
$+\ \{4(\overline{4};21)\}$	$+\ \{4(\overline{4};3)\}$	$+\ \{4(\overline{41};2^2)\}$	$+\ 2\{4(\overline{41};31)\}$	$+\ 2\{4(\overline{41};4)\}$
$+\ \{4(\overline{41^2};5)\}$	$+\ 2\{4(\overline{42};32)\}$	$+\ 2\{4(\overline{42};41)\}$	$+\ 2\{4(\overline{42};5)\}$	$+\ \{4(\overline{5};21^2)\}$
$+\ \{4(\overline{5};2^2)\}$	$+\ 2\{4(\overline{5};31)\}$	$+\ \{4(\overline{5};4)\}$	$+\ 2\{4(\overline{51};32)\}$	$+\ 3\{4(\overline{51};41)\}$
$+\ 2\{4(\overline{51};5)\}$	$+\ \{4(\overline{6};2^21)\}$	$+\ \{4(\overline{6};31^2)\}$	$+\ 2\{4(\overline{6};32)\}$	$+\ 2\{4(\overline{6};41)\}$
$+\ \{4(\overline{6};5)\}$				

Result for $U(2,2)$

$\{4(\overline{31};21)\}$	$+\ \{4(\overline{31};3)\}$	$+\ \{4(\overline{32};2^2)\}$	$+\ \{4(\overline{32};31)\}$	$+\ \{4(\overline{32};4)\}$
$+\ \{4(\overline{3^2};32)\}$	$+\ \{4(\overline{3^2};41)\}$	$+\ \{4(\overline{3^2};5)\}$	$+\ \{4(\overline{4};21)\}$	$+\ \{4(\overline{4};3)\}$
$+\ \{4(\overline{41};2^2)\}$	$+\ 2\{4(\overline{41};31)\}$	$+\ 2\{4(\overline{41};4)\}$	$+\ 2\{4(\overline{42};32)\}$	$+\ 2\{4(\overline{42};41)\}$
$+\ 2\{4(\overline{42};5)\}$	$+\ \{4(\overline{5};2^2)\}$	$+\ 2\{4(\overline{5};31)\}$	$+\ \{4(\overline{5};4)\}$	$+\ 2\{4(\overline{51};32)\}$
$+\ 3\{4(\overline{51};41)\}$	$+\ 2\{4(\overline{51};5)\}$	$+\ 2\{4(\overline{6};32)\}$	$+\ 2\{4(\overline{6};41)\}$	$+\ \{4(\overline{6};5)\}$

Note that we have inserted the integer 4 into each irrep to yield the correct description of the irreps.

8. A SCHUR function

The previous example illustrates the general problems involved in determining Kronecker products for a pair of harmonic series irreps of $U(p,q)$. Much of the preceding work can be avoided by defining a function that can be read into SCHUR and run for any pair of irreps. Such a function can be written as:-

```
gr4u4u4u4u4
sb_rev true
setlimit6
set_pwt6
enter v1
enter v2
setp3[0*0*0*0]+[0*0*1*1]+[0*0*2*2]+[0*0*11*11]+[0*0*3*3]+[0*0*21*21]
setp3add v3,[0*0*4*4]+[0*0*31*31]+[0*0*22*22]+[0*0*5*5]+[0*0*41*41]
setp3add v3,[0*0*32*32]+[0*0*6*6]+[0*0*51*51]+[0*0*42*42]+[0*0*33*33]
wt1,6p v1,v2
len1,2last
len2,2last
wt1,6p last,v3
```

```
wt1,6contract1,3last
wt1,6contract2,3last
len1,3last
len2,3last
setp1last
len1,1last
add len2,1v1,last
setp2last
len1,-3last
add len2,-3v2,last
setp3last
len1,2v1
len2,2last
add last,v3
setp3last
columns5
sb_tex true
last
stop
```

and saved as "upq.fn" and run as follows:

```
DP>
readfn1'upq.fn'
=-
=-
=-
=-
=-
=-
=-
=-
=-
=-
=-
=-
=-
=-
=-
=-
DP>
fn1
Groups are   U(4) * U(4) * U(4) * U(4)
enter v1
[3*1*0*0]-[31*11*0*0]
```

```
enter v2
[1*2*0*0]-[11*21*0*0]
Groups are   U(4) * U(4) * U(4)
Groups are   U(4) * U(4)
      \+$\{31\}\{21\}$&$ + \ \{31\}\{3\}$&$ + \ \{31^2\}\{4\}$&
$ + \ \{32\}\{2^2\}$&$ + \ \{32\}\{31\}$\cr
      \+$ + \ \{32\}\{4\}$&$ + \ \{321\}\{5\}$&$ + \ \{3^2\}\{32\}$&
$ + \ \{3^2\}\{41\}$&$ + \ \{3^2\}\{5\}$\cr
      \+$ + \ \{4\}\{21\}$&$ + \ \{4\}\{3\}$&$ + \ \{41\}\{2^2\}$&
$ + \ 2\{41\}\{31\}$&$ + \ 2\{41\}\{4\}$\cr
      \+$ + \ \{41^2\}\{5\}$&$ + \ 2\{42\}\{32\}$&$ + \ 2\{42\}\{41\}$&
$ + \ 2\{42\}\{5\}$&$ + \ \{5\}\{21^2\}$\cr
      \+$ + \ \{5\}\{2^2\}$&$ + \ 2\{5\}\{31\}$&$ + \ \{5\}\{4\}$&
$ + \ 2\{51\}\{32\}$&$ + \ 3\{51\}\{41\}$\cr
      \+$ + \ 2\{51\}\{5\}$&$ + \ \{6\}\{2^21\}$&$ + \ \{6\}\{31^2\}$&
$ + \ 2\{6\}\{32\}$&$ + \ 2\{6\}\{41\}$\cr
      \+$ + \ \{6\}\{5\}$\cr
DP>
```

To produce the same final results. The advantage of such a function is that it can be used repeatedly, one simply enters the relevant signed sequences. One may also change the groups set to obtain results for various $U(p,q)$.

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