Lecture Notes on Symmetry 1994

Proposition: *We should always strive to construct theories with the highest possible symmetry even if these are not exact symmetries of nature. The physics comes in the process of breaking the symmetry.*

Consider the case of

$$Ce_2Mg_3(NO_3)_{12} \cdot 24H_2O : Nd^{3+}$$

- **What symmetry does the Nd^{3+} ion see in the rare earth double nitrates?**
- **The entire breakdown of the symmetry could be described by the chain of nested subgroups**

$$SO_3 \supset K_h \supset T_h \supset C_3$$

- **Global and Local Symmetries**

A symmetry may be *global* or *local*.

- **Types of Symmetries**
- **Discrete symmetries, such as reflections, inversions, time reversal, charge conjugation, parity, finite rotations, permutations etc. are associated with *multiplicative* or *phase-like* quantum numbers.**
- **Continuous symmetries such as translations and rotations are associated with *additive* quantum numbers (e.g. angular momentum J or linear momentum p).**

- Bosons and Fermions
- The particles we commonly encounter in physics can be divided into two classes *bosons* and *fermions*. Bosons are associated with *integer* spin, examples being photons, gluons and the weak interaction bosons Z^0 and $W^{\pm}$. Fermions are associated with *half-integer* spin, examples being electrons, neutrinos and quarks. Bosons establish the *interactions* between fermions. Thus the *photon*, a massless spin 1 particle, is the exchange particle associated with electromagnetic interactions. In most of atomic and molecular physics we can restrict our attention to quantum electrodynamics (QED). The weak interactions manifest themselves in atomic and molecular physics in very small parity violations. Bosons and fermions obey different statistics, namely Bose-Einstein and Fermi-Dirac, respectively. That requires us to construct totally symmetric wavefunctions for many-boson systems and totally antisymmetric wavefunctions for many-fermion systems.
- Permutational Symmetry

Bosons and fermions differ with respect to their behaviour under an interchange of their position, or equivalently with respect to a rotation through 2π or 360^o. We shall designate the wavefunction for a single fermion or boson as $\phi(\alpha)$ where α is an appropriate set of single particle quantum numbers associated with some single particle solution of , for example, some central field potential. Thus for a hydrogen atom we might use $\alpha = \{ns\ell m_s m_\ell\}$ or $\alpha = \{ns\ell j m_j\}$.

A $N-$particle system will involve $N-$single particle wavefunctions ($\phi_i \qquad i = 1, 2, \ldots, N$) and $N-$sets of single particle quantum numbers ($\alpha_k \qquad k = 1, 2, \ldots, N$). The wavefunction, Ψ, for the $N-$particle system will be such that

$$\Psi = \Psi(\phi_1, \phi_2, \ldots, \phi_N) \tag{1.1}$$

For a two-particle system we could write

$$\Psi(\phi_1, \phi_2) = \frac{1}{\sqrt{2}}\{\phi_1(\alpha_1)\phi_2(\alpha_2) \pm \phi_1(\alpha_2)\phi_2(\alpha_1)\} \tag{1.2}$$

- **The** *positive* **sign corresponds to a** *symmetric* **wavefunction and the** *minus* **sign corresponds to an** *antisymmetric* **wavefunction. Note that we have permuted the quantum numbers with respect to the coordinates of the particles. The wavefunction of a pair of fermions, unlike a pair of bosons, undergoes a change of sign. If $\alpha_1 = \alpha_2$ then for identical fermions Eq.(1.2) vanishes though not for bosons. That is consistent with the Pauli exclusion principle for identical fermions.**
- **Thus permutational symmetry, required by the indistinguishability of identical particles, leads for $N-$fermions to the construction of of** *determinantal states* **to give totally** *antisymmetric* **states while for $N-$bosons to the construction of** *permanental states* **to give totally** *symmetric* **states.**

- **Hence for an $N-$fermion system we have the totally antisymmetric wavefunction**

$$\Psi(\phi_1,\phi_2,\ldots,\phi_N) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \phi_1(\alpha_1) & \phi_1(\alpha_2) & \ldots & \phi_1(\alpha_N) \\ \phi_2(\alpha_1) & \phi_2(\alpha_2) & \ldots & \phi_2(\alpha_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(\alpha_1) & \phi_N(\alpha_2) & \ldots & \phi_N(\alpha_N) \end{vmatrix}^{\{1^N\}} \quad (1.3)$$

- **In $LS-$coupling basis we use $\alpha = \{ns\ell m_s m_\ell\}$ whereas in $jj-$coupling we would use $\alpha = \{ns\ell j m_j\}$.**
- **The information content of the determinantal state may be fully specified by the abbreviated form**

$$\{\alpha_1\alpha_2\ldots\alpha_N\} \quad (1.4)$$

- **In the case of bosons we are required to construct permanental states to yield totally symmetric wavefunctions,**

$$\Psi(\phi_1,\phi_2,\ldots,\phi_N) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \phi_1(\alpha_1) & \phi_1(\alpha_2) & \ldots & \phi_1(\alpha_N) \\ \phi_2(\alpha_1) & \phi_2(\alpha_2) & \ldots & \phi_2(\alpha_N) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(\alpha_1) & \phi_N(\alpha_2) & \ldots & \phi_N(\alpha_N) \end{vmatrix}^{\{N\}} \quad (1.5)$$

- **The information content of the permanental state may be fully specified by the abbreviated form**

$$[\alpha_1\alpha_2\ldots\alpha_N] \quad (1.6)$$

- **Many-particle states of Bosons and Fermions**

Let us for the moment consider the states of N identical bosons or fermions. Suppose the boson has an angular momentum $j = 2$ (i.e. a $d-$boson) and hence $m_j = 0, \pm 1, \pm 2$ while the fermion has angular momentum $j = 5/2$ and hence $m_j = \pm 1/2, \pm 3/2, \pm 5/2$. If $N = 2$ in both cases what are the allowed values of J ? We note that

$$M_J = m_{j_1} + m_{j_2}$$

Just considering the non-negative values of M_J we obtain for the fermions the following table of determinantal states:

Table 1.1 Determinantal states for $(5/2)^2$ fermions.

M_J	States		
4	$\{5/2\,3/2\}$		
3	$\{5/2\,1/2\}$		
2	$\{5/2\ -1/2\}$	$\{3/2\,1/2\}$	
1	$\{5/2\ -3/2\}$	$\{3/2\ -1/2\}$	
0	$\{5/2\ -5/2\}$	$\{3/2\ -3/2\}$	$\{1/2\ -1/2\}$

- **Inspection of the above table leads to the conclusion that the allowed values of J in $(5/2)^2$ are $J = 0, 2, 4$.**

The corresponding d^2 boson states for non-negative M_J are given in Table 1.2.

Table 1.2. Permanental states for d^2 bosons

M_J		States	
4	$[2\,2]$		
3	$[2\,1]$		
2	$[2\,0]$	$[1\,1]$	
1	$[2\ -1]$	$[1\,0]$	
0	$[2\ -2]$	$[1\ -1]$	$[0\,0]$

Inspection of the above table leads to the conclusion that the allowed values of J in d^2 are $J = 0, 2, 4$ exactly those found for $(5/2)^2$.

Exercises

1.1 Show that the totally antisymmetric orbital angular momentum states of g^3 $(\ell = 4)$ (i.e. the states of maximum multiplicity) are the same as for the totally symmetric states of $(5/2)^4$.

1.2 Determine the allowed values of J for the $jj-$coupled configurations $(5/2)^2, (5/2\,7/2)$ and $(7/2)^2$.

1.3 Determine the allowed values of S and L for the electron configuration f^2.

1.4 Given that for an $LS-$coupled term ^{2S+1}L we have $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and

$$J = L + S, L + S - 1, \ldots, |L - S| \qquad (1.7)$$

Show that the values of J for the list of terms found in Ex 1.3. are the same as those found in Ex 1.2.

1.5 **Show that in the configuration j^2 the only allowed values of J are the even integers $0, 2, \ldots, 2j-1$.**

1.6 **Starting with the angular momentum commutation relations**

$$[J_x, J_y] = iJ_z \quad , [J_y, J_z] = iJ_x \quad , [J_z, J_x] = iJ_y \tag{1.8}$$

show that if $J_\pm = J_x \pm iJ_y$ then

$$\mathbf{J}^2 = \frac{J_+J_- + J_-J_+}{2} + J_z^2 \tag{1.9}$$

1.7 **If $\mathbf{J} = \mathbf{L} + \mathbf{S}$ show that**

$$J(J+1) - L(L+1) - S(S+1) = S_+L_- + S_-L_+ + 2S_zL_z \tag{1.10}$$

- **Ladder Operators and Determinantal States**

For the electron configuration f^2 we can enumerate the set of determinantal states for non-negative M_S, M_L as in Table 2.1.

Table 2.1. Determinantal states for the Electron Con-

figuration f^2.

M_L	$M_S = 0$				$M_S = 1$	
6	$\left\{\begin{smallmatrix}+ & -\\ 3 & 3\end{smallmatrix}\right\}$					
5	$\left\{\begin{smallmatrix}+ & -\\ 3 & 2\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 2 & 3\end{smallmatrix}\right\}$			$\left\{\begin{smallmatrix}+ & +\\ 3 & 2\end{smallmatrix}\right\}$	
4	$\left\{\begin{smallmatrix}+ & -\\ 3 & 1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 2 & 2\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 1 & 3\end{smallmatrix}\right\}$		$\left\{\begin{smallmatrix}+ & +\\ 3 & 1\end{smallmatrix}\right\}$	
3	$\left\{\begin{smallmatrix}+ & -\\ 3 & 0\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 2 & 1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 1 & 2\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 0 & 3\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & +\\ 3 & 0\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & +\\ 2 & 1\end{smallmatrix}\right\}$
2	$\left\{\begin{smallmatrix}+ & -\\ 2 & 0\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 3 & -1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 1 & 1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 0 & 2\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & +\\ 3 & -1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & +\\ 2 & 0\end{smallmatrix}\right\}$
	$\left\{\begin{smallmatrix}+ & -\\ -1 & 3\end{smallmatrix}\right\}$					
1	$\left\{\begin{smallmatrix}+ & -\\ 1 & 0\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 2 & -1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 3 & -2\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 0 & 1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & +\\ 3 & -2\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & +\\ 1 & 0\end{smallmatrix}\right\}$
	$\left\{\begin{smallmatrix}+ & -\\ -1 & 2\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ -2 & 3\end{smallmatrix}\right\}$			$\left\{\begin{smallmatrix}+ & +\\ 2 & -1\end{smallmatrix}\right\}$	
0	$\left\{\begin{smallmatrix}+ & -\\ 0 & 0\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 1 & -1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 2 & -2\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ 3 & -3\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & +\\ 3 & -3\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & +\\ 2 & -2\end{smallmatrix}\right\}$
	$\left\{\begin{smallmatrix}+ & -\\ -1 & 1\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ -3 & 3\end{smallmatrix}\right\}$	$\left\{\begin{smallmatrix}+ & -\\ -2 & 2\end{smallmatrix}\right\}$		$\left\{\begin{smallmatrix}+ & +\\ 1 & -1\end{smallmatrix}\right\}$	

- **For an electron in an $f-$orbital $\ell = 3$ hence $m_\ell = 0, \pm 1, \pm 2, \pm 3$. There are just two values of the spin projection $m_s = \pm 1/2$.**
- **In writing a determinantal state it suffices to just display the values of m_ℓ and indicate the value of m_s as a $+$ or $-$ sign placed above m_ℓ.**
- **For a given determinantal state**

$$M_S = \sum_{i=1}^{n} m_{s_i} \qquad \text{and} \qquad M_L = \sum_{i=1}^{n} m_{\ell_i} \tag{2.1}$$

- **Every determinantal state may be associated with definite values of M_S and M_L.**
- **Form appropriate linear combinations of the determinantal states to give eigenstates $|SLM_SM_L>$.**

- **Write a state as $|^{2S+1}LM_SM_L>$ where $(2S+1)$ is the *spin multiplicity*.**
- **The quantum number L is associated with alphabetical letters**

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ S & P & D & F & G & H & I & K & M \end{array}$$

- **A spectroscopic term will be designated as**

$$^{2S+1}L$$

- **Associated with a given value on S there are $(2S+1)$ values of M_S and with L there are $(2L+1)$ values of M_L where**

$$\begin{aligned} M_S &= S, S-1, \ldots, -S+1, -S \\ M_L &= L, L-1, \ldots, -L+1, -L \end{aligned}$$

Inspection of Table 2.1 shows that the spectroscopic terms of the electron configuration f^2 are

$$^3PFH \quad ^1SDGI$$

Choose

$$|^1I06> \equiv \{\begin{smallmatrix} + & - \\ 3 & 3 \end{smallmatrix}\} \tag{2.2}$$

Let us now determine $|^1I05>$. To do this we use the properties of ladder operators. Recall

$$L_{\pm}|LM> = \sqrt{L(L+1) - M(M \pm 1)}|LM \pm 1> \tag{2.3}$$

and

$$L_{\pm} = \sum_{i=1}^{n} \ell_{\pm_i} \tag{2.4}$$

Let (2.3) act on the left-hand-side of (2.2) and noting (2.3) act also on the determinantal state to give

$$L_{-}|^1I06> = \sqrt{6\times 7 - 6\times 5}|^1I05> = \sqrt{12}|^1I05> \tag{2.5}$$

and

$$L_{-}\{\begin{smallmatrix} + & - \\ 3 & 3 \end{smallmatrix}\} = \sqrt{3\times 4 - 3\times 2}[\{\begin{smallmatrix} + & - \\ 2 & 3 \end{smallmatrix}\} + \{\begin{smallmatrix} + & - \\ 3 & 2 \end{smallmatrix}\}] \tag{2.5}$$

Equating (2.4) and (2.5) gives

$$|^1I05> = \frac{\sqrt{2}}{2}[\{\begin{smallmatrix} + & - \\ 2 & 3 \end{smallmatrix}\} + \{\begin{smallmatrix} + & - \\ 3 & 2 \end{smallmatrix}\}] \tag{2.6}$$

This state must be orthogonal to the state $|^3H05>$ and hence after fixing a phase we have

$$|^3H05> = \frac{\sqrt{2}}{2}[\{\begin{smallmatrix} + & - \\ 2 & 3 \end{smallmatrix}\} - \{\begin{smallmatrix} + & - \\ 3 & 2 \end{smallmatrix}\}] \tag{2.7}$$

Application of the spin raising operator S_+ to (2.7) gives

$$S_+|^3H05> = \sqrt{2}|^3H15> = \frac{\sqrt{2}}{2}[\{\begin{smallmatrix} + & + \\ 2 & 3 \end{smallmatrix}\} - \{\begin{smallmatrix} + & + \\ 3 & 2 \end{smallmatrix}\}] \tag{2.7a}$$

and hence

$$|^3H15> = -\{\begin{smallmatrix} + & + \\ 3 & 2 \end{smallmatrix}\}] \tag{2.8}$$

Note the appearance of the minus sign which comes from our particular choice of enumeration of the determinantal states.

Exercises

2.1 Determine the eigenstates

$$|^1I04> \quad |^3H04> \quad |^1G04> \quad |^3H14>$$

as linear combinations of determinantal states.

2.2 Discuss how you could determine the eigenstates $|^3HJM>$ as linear combinations of the states $|^3HM_SM_L>$. *Hint***:** use the fact that $J_\pm = L_\pm + S_\pm$**.**

- **Permutations and the Symmetric Group**

Permutations play an important role in the physics of identical particles. A permutation leads to a reordering of a sequence of objects. We can place n objects in the natural number ordering $1, 2, \ldots, n$. Any other ordering can be discussed in terms of this ordering and can be specified in a two line notation

$$\begin{matrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{matrix} \tag{2.9}$$

For $n = 3$ we have the six permutations

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tag{2.10}$$

Permutations can be multiplied working from right to left. Thus

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

The six permutations in (2.10) satisfy the following properties:

1. **There is an identity element** $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$**.**
2. **Every element has an inverse among the set of elements.**
3. **The product of any two elements yields elements of the set.**
4. **The elements satisfy the associativity condition a(bc) = (ab)c.**

These conditions establish that the permutations form a group. In general the $n!$ **permutations form the elements of the symmetric group** $\mathcal{S}_n$**.**

- **Cycle Structure of Permutations**

It is useful to express permutations as a cycle structure. A cycle $(i, j, k, \ldots, l)$ **is interpreted as** $i \to j$, $j \to k$ **and finally** $l \to i$**. Thus our six permutations have the cycle structures**

$$(1)(2)(3), (1,2)(3), (1)(2,3), (1,3)(2), (1,3,2), (1,2,3) \quad (2.11)$$

The elements within a cycle can be cyclically permuted and the order of the cycles is irrelevant. Thus $(123)(45) \equiv (54)(312)$**.**

- **A** $k-$*cycle* **or** *cycle of length* k **contains** k **elements. It is useful to organise cycles into** *types* **or** *classes***.**

We shall designate the *cycle type* of a permutation π by

$$(1^{m_1}\, 2^{m_2}\, \ldots, n^{m_n}) \tag{2.12}$$

where m_k is the number of cycles of length k in the cycle representation of the permutation π.

- **For $\mathcal{S}_4$ there are five cycle types**

$$(1^4),\ (1^2\, 2^1),\ (2^2),\ (1^1\, 3^1),\ (4^1) \tag{2.13}$$

Normally exponents of unity are omitted and Eq.(2.13) written as

$$(1^4),\ (1^2\, 2),\ (2^2),\ (1\, 3),\ (4) \tag{2.14}$$

- **Cycle types may be equally well labelled by ordered partitions of the integer n**

$$\lambda = (\lambda_1\, \lambda_2\, \ldots\, \lambda_\ell) \tag{2.15}$$

where the λ_i are weakly decreasing and

$$\sum_{i=1}^{\ell} \lambda_i = n \tag{2.16}$$

The partition is said to be of *length* ℓ and of *weight* n. In terms of partitions the cycle types for $\mathcal{S}_5$ are

$$(1^5),\ (2\, 1^3),\ (2^2\, 1),\ (3\, 2),\ (3\, 1^2),\ (4\, 1),\ (5)$$

- **Conjugacy Classes of $\mathcal{S}_n$**

In any group G the elements g and h are *conjugates* if

$$g = k\,h\,k^{-1} \qquad \textbf{for some} \qquad k \in G \tag{2.17}$$

The set of all elements conjugate to a given g is call the *conjugacy class* of g which we denote as K_g.

Exercises

2.3 **Show that for $\mathcal{S}_4$ there are five conjugacy classes that may be labelled by the five partitions of the integer 4.**

2.4 **Show that the permutations, expressed in cycles with cycles of length one suppressed, divide among the conjugacy classes as**

$$\begin{aligned}
(1^4) &\supset e \\
(2\,1^2) &\supset (1\,2),\ (1\,3),\ (1\,4),\ (2\,3),\ (2\,4),\ (3\,4) \\
(2^2) &\supset (1\,2)(3\,4),\ (1\,3)(2\,4),\ (1\,4)(2\,3) \\
(3\,1) &\supset (1\,2\,3),\ (1\,2\,4),\ (1\,3\,2),\ (1\,3\,4),\ (1\,4\,2) \\
&\quad\ (1\,4\,3),\ (2\,3\,4),\ (2\,4\,3) \\
(4) &\supset (1\,2\,3\,4),\ (1\,2\,4\,3),\ (1\,3\,4\,2),\ (1\,4\,3\,2)
\end{aligned} \tag{2.18}$$

In general two permutations are in the same conjugacy class if, and only if, they are of the same cycle type. The number of classes of $\mathcal{S}_n$ is equal the number of partitions of the integer n.
If $\lambda = (1^{m_1}\,2^{m_2}\,\ldots\,n^{m_n})$ then the number of permutations k_λ in the class (λ) of $\mathcal{S}_n$ is

$$k_\lambda = \frac{n!}{1^{m_1}\,m_1!\,2^{m_2}\,m_2!\,\ldots\,n^{m_n}\,m_n!} \tag{2.19}$$

- **The Alternating Group $\mathcal{A}_n$**

1. **A cycle of order two is termed a** *transposition*.
2. **A transposition $(i,\, i+1)$ is termed an** *adjacent transposition*.
3. **The entire symmetric group $\mathcal{S}_n$ can be generated (or given a** *presentation* **in terms of the set of adjacent transpositions**

$$(1\,2),\,(2\,3),\,\ldots,\,(n-1\,n) \tag{2.20}$$

- **If $\pi = \tau_1\tau_2\ldots\tau_k$, where the τ_i are transpositions then the** *sign* **of π is defined to be**

$$sgn(\pi) = (-1)^k \tag{2.21}$$

If the number of cycles of *even* **order is** *even* **then the permutation is** *even* **or** *positive*; **if it is** *odd* **then the permutation is** *odd* **or** *negative*.

The set of *even* **permutations form a subgroup of $\mathcal{S}_n$ known as the** *alternating group* $\mathcal{A}_n$ **and has precisely half the elements of $\mathcal{S}_n$ i.e. $(\frac{1}{2})n!$.**

Exercises

2.5 Show that the set of six matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \tag{2.22}$$

with the usual rule of matrix multiplication form a group isomorphic to $\mathcal{S}_3$.

2.6 Show that the symmetric group $\mathcal{S}_n$ has two one-dimensional representations, a symmetric representation where every element is mapped onto unity and an antisymmetric representation where the elements are mapped onto the sign defined in Eq. (2.21).

Lecture Three

Partitions

- A partition is any finite or infinite sequence of integers

$$\lambda = (\lambda_1\, \lambda_2\, \ldots\, \lambda_i\, \ldots) \tag{3.1}$$

Unless otherwise stated we shall assume the sequence involves non-negative integers in non-increasing order;

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_i \geq \ldots \tag{3.2}$$

Normally we will omit zeros.

- The non-zero λ_i form the *parts* of λ. The number of parts is the *length*, $\ell(\lambda)$, of λ while the sum of its parts, $|\lambda|$, is the *weight* of λ. If $|\lambda| = n$ then λ is said to be a *partition* of n.
- We shall frequently write $\lambda \vdash n$ to indicate that λ is a partition of n. Repeated parts of a partition will frequently be indicated as i^{m_i} where m_i is the number of times the part i occurs in the partition λ.
- The partitions for $n = 6$ are

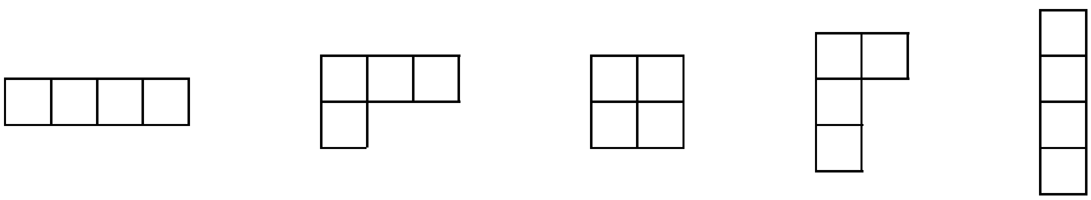
$$(6)\,(51)\,(42)\,(41^2)\,(3^2)\,(321)\,(31^3)\,(2^3)\,(2^21^2)\,(21^4)\,(1^6)$$

- Note, in the above example the partitions have been listed in *reverse lexicographic order*. The ordering is such that the first non-vanishing difference $\lambda_i - \mu_i$, for successive partitions λ, μ is *positive*.

The Ferrers-Sylvester diagram

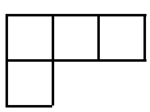
- Every partition $\lambda \vdash n$ may be associated with a *Ferrers-Sylvester diagram*, *shape* or *frame* involving n cells, dots or boxes in $\ell(\lambda)$ left-adjusted rows with the i-th row containing λ_i cells, dots, or boxes.

For $n = 4$ we have the five diagrams



- We will formally designate the frame associated with a partition λ as F^{λ}.
- The *conjugate* of a partition λ is a partition λ' whose diagram is the transpose of the diagram of λ. If $\lambda' \equiv \lambda$ then the partition λ is said to be *self-conjugate*.

Thus



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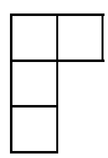
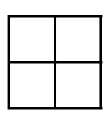


are conjugates while

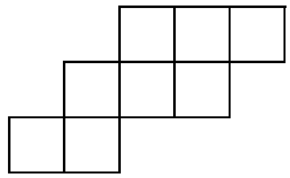


is self-conjugate.

Skew frames

- Given two partitions λ and μ such that $\lambda \supset \mu$ implies that the frame F^λ contains the frame F^μ, i.e. that $\lambda_i \geq \mu_i$ for all $i \geq 1$. The difference $\rho = \lambda - \mu$ forms a *skew* frame $F^{\lambda/\mu}$.

Thus, for example, the skew frame $F^{542/21}$ has the form



Note that a skew frame may consist of disconnected pieces.

Frobenius notation for partitions

- There is an alternative notation for partitions due to Frobenius. The *diagonal* of nodes in a Ferrers-Sylvester diagram beginning at the top left-hand corner is called the *leading diagonal*. The number of nodes in the leading diagonal is called the *rank* of the partition.
- If r is the rank of a partition then let a_i be the number of nodes to the right of the leading diagonal in the i-th row and let b_i be the number of nodes below the leading diagonal in the i-th column. The partition is then denoted by Frobenius as

$$\begin{pmatrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_r \end{pmatrix} \tag{3.3}$$

We note that

$$a_1 > a_2 > \ldots > a_r$$
$$b_1 > b_2 > \ldots > b_r$$

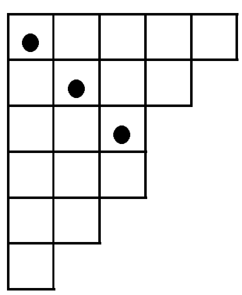
and

$$a_1 + a_2 + \ldots + a_r + b_1 + b_2 + \ldots + b_r + r = n$$

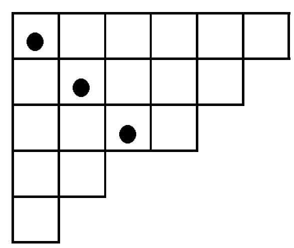
- The partition conjugate to that of Eq.(3.3) is just

$$\begin{pmatrix} b_1, & b_2, & \ldots, & b_r \\ a_1, & a_2, & \ldots, & a_r \end{pmatrix} \tag{3.4}$$

As an example consider the partitions $(5\,4\,3^2\,2\,1)$ and $(6\,5\,4\,2\,1)$. Drawing their diagrams and marking their leading diagonal we have



and



from which we deduce the respective Frobenius designations

$$\begin{pmatrix} 4 & 2 & 0 \\ 5 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

Young tableaux

- A Young tableau is an assignment of n numbers to the n cells of a frame F^λ with $\lambda \vdash n$ according to some numbering sequence.

- A tableau is *standard* if the assignment of the numbers $1, 2, \ldots, n$ is such that the numbers are positively increasing from left to right in rows and down columns from top to bottom.

Thus for the partitions of the integer 4 we have the standard Young tableaux

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$$

- In the above examples the number of standard tableaux for conjugate partitions is the same. Indeed the number of standard tableaux associated with a given frame F^{λ} is the *dimension* f_n^{λ} of an irreducible representation $\{\lambda\}$ of the symmetric group $\mathcal{S}_n$.

Hook lengths and dimensions for $\mathcal{S}_n$

- **The** *hook length* **of a given box in a frame F^λ is the length of the right-angled path in the frame with that box as the upper left vertex.**

For example, the hook length of the marked box in

$$\begin{array}{|c|c|c|c|c|}
\hline
 & & & & \\
\hline
* & \cdot & \cdot & \cdot & \multicolumn{1}{|c}{} \\
\cline{1-4}
\cdot & & & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\
\cline{1-3}
\cdot & & & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\
\cline{1-3}
\cdot & & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-2}
\cdot & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}$$

is 8.

Theorem 3.1: *To find the dimension of the representation of $\mathcal{S}_n$ corresponding to the frame F^λ, divide $n\,!$ by the factorial of the hook length of each box in the first column of F^λ and multiply by the difference of each pair of such hook lengths.*

Thus for the partition $(5\,4\,3^2\,2\,1)$ we have the hook lengths

$$\begin{array}{|c|c|c|c|c|}
\hline
10 & & & & \\
\hline
8 & & & & \multicolumn{1}{|c}{} \\
\cline{1-4}
6 & & & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\
\cline{1-3}
5 & & & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\
\cline{1-3}
3 & & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-2}
1 & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}$$

and hence a dimension

$$\begin{aligned}
f_{18}^{543^2 21} &= 18\,!\frac{2\times4\times5\times7\times9\times2\times3\times5\times7\times1\times3\times5\times2\times4\times2}{10\,!\times8\,!\times6\,!\times5\,!\times3\,!\times1\,!} \\
&= 10720710
\end{aligned}$$

It is not suggested that you check the above result by explicit enumeration!

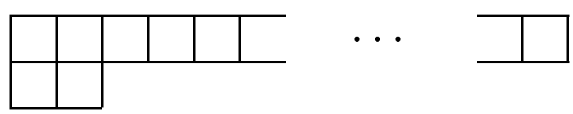
- The above evaluation can also be equivalently made by computing the hook lengths h_{ij} for every box at position (i,j) and then noting that

$$f_n^\lambda = \frac{n\,!}{\prod_{(i,j)\in\lambda} h_{ij}} \tag{3.5}$$

  which is the celebrated result of Frame, Robinson and Thrall.

Exercises

3.1 Show that the dimension of of the representation

$\{p{+}2, 2\} =$ 

is

$$\frac{1}{2}(p{+}4)(p{+}1)$$

3.2 Calculate the dimensions of the irreducible representations of $\mathcal{S}_6$ and show that

$$\sum_{\lambda\vdash 6} (f_6^\lambda)^2 = 6\,!$$

The symmetric group and tensors

- Let $T_{\mu_1 \ldots \mu_n}$ be a "generic" n-index tensor, without any special symmetry. (For the moment, "tensor" means just a function of n indices, not necessarily with any geometrical realization. It must be meaningful, however, to *add* (and form linear combinations of) tensors of the same rank.)
- The entries $1, 2, \ldots n$ in the standard numbering of a tableau indicate the n successive indices of $T_{\mu_1 \cdots \mu_n}$.
- The tableau defines a certain symmetrization operation on these indices: *symmetrize* on the set of indices indicated by the entries in each row, then *antisymmetrize* the result on the set of indices indicated by the entries in each column.
- The resulting object is a tensor, T, with certain index symmetries. Now let each permutation in S_n act (separately) upon T. The $n!$ results are not linearly independent; they span a vector space which supports an irreducible representation of S_n.
- Different tableaux corresponding to the same frame yield equivalent (but not identical) representations.

Example: The partition $\{2\,2\}$ of 4 has two standard tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \textit{and} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \tag{3.6}$$

Let us construct the symmetrized tensor T corresponding to the second of these.

- **First symmetrize over the first and third indices, and over the second and fourth:**

$$\frac{1}{4}\big(T_{abcd} + T_{cbad} + T_{adcb} + T_{cdab}\big).$$

Now antisymmetrize the result over the first and second indices, and the third and fourth; dropping the combinatorial factor $\frac{1}{16}$, we get

$$\begin{aligned} \mathcal{T}_{abcd} = {} & T_{abcd} + T_{cbad} + T_{adcb} + T_{cdab} \\ & - T_{bacd} - T_{cabd} - T_{bdca} - T_{cdba} \\ & - T_{abdc} - T_{dbac} - T_{acdb} - T_{dcab} \\ & + T_{badc} + T_{dabc} + T_{bcda} + T_{dcba} \,. \end{aligned} \tag{3.7}$$

It is easy (though tedious) to check that $\mathcal{T}$ possesses the symmetries characteristic of the Riemann tensor.

Exercise

3.3 Construct a set of three 4-index tensors corresponding to the three Young tableaux associated with the partition $\{3\,1\}$.

Unitary numbering of Young tableaux

- **Many different prescriptions can be given for injecting numbers into the boxes of a frame.**
- **The standard numbering is intimately associated with the symmetric group $\mathcal{S}_n$.**
- **Another important numbering prescription is that of *unitary* numbering where now numbers 1, 2, ..., d are injected into the boxes of a frame F^λ such that:**

i. Numbers are non-decreasing across a row going from left to right.

ii. Numbers are positively increasing in columns from top to bottom.

• The first condition permits repetitions of integers. Using the numbers $1, 2, 3$ **in the frame** $F^{2\,1}$ **we obtain the 8 tableaux**

$$\begin{array}{ccccc} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & & \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \end{array} \tag{3.8}$$

Had we chosen $d = 2$ **we would have obtained just two tableaux while** $d = 4$ **yields twenty tableaux. In general, for a frame** F^λ **a unitary numbering using the integers** $1, 2, \ldots, d$ **leads to**

$$f_d^\lambda = \frac{G_d^\lambda}{H_\lambda} \tag{3.9}$$

where H_λ **is the product of the hook lengths** h_{ij} **of the frame and**

$$G_d^\lambda = \prod_{(i,j)\in\lambda} (d+i-j) \tag{3.10}$$

Thus for $d = 5$ **and** $\lambda = (4\,2\,1)$ **we have** $H_{(4\,2\,1)} = 144$ **and** $G_5^{\{4\,2\,1\}} = 100800$ **from which we deduce that**

$$f_5^{\{4\,2\,1\}} = 700$$

which is the dimension of the irreducible representation $\{4\,2\,1\}$ **of the general linear group** $GL(5)$**.**

- In general, f_d^λ is the dimension of the irreducible representation $\{\lambda\}$ of $GL(d)$. Since the representations of $GL(d)$ labelled by partitions λ remain irreducible under restriction to the unitary group $U(d)$ **Eq.(3.9)** is valid for computing the dimensions of the irreducible representations of the unitary group $U(d)$.
- The same rules for a unitary numbering may be applied to the skew frames $F^{\lambda/\mu}$ introduced in §**3.3.** Thus for $F^{542/21}$ an allowed unitary numbering using just the integers 1 and 2 would be

$$\begin{array}{ccccc} & & \boxed{1} & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \boxed{2} & \\ \boxed{1} & \boxed{2} & & & \end{array}$$

- Note that our unitary numbering yields what in the mathematical literature are commonly referred to as *semistandard* Young tableaux. Other numberings are possible and have been developed for all the classical Lie algebras.

Exercises

3.4 Draw the frames $F^{2^2/1}$, $F^{43^21/421^2}$, and $F^{321/21}$.

3.5 Use the integers 1, 2, 3 to construct the complete set of semistandard tableaux for the frame $F^{43^21/421^2}$ and show that the same number of tableaux arise for the frame F^{21}.

3.6 Make a similar unitary numbering for the frame $F^{321/21}$ and show that the same number of semistandard tableaux arise in the set of frames $F^3 + 2F^{21} + F^{1^3}$.

Lecture Four

Young tableaux and monomials

A numbered frame may be associated with a unique monomial by replacing each integer i by a variable x_i. Thus the Young tableau

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 5 \\ \hline 3 & 3 & 3 & 5 \\ \cline{1-4} 4 & 6 & 7 \\ \cline{1-3} 5 & 7 & 8 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

can be associated with the monomial
$x_1^2\, x_2\, x_3^3\, x_4^2\, x_5^3\, x_6^2\, x_7^3\, x_8^2$

Monomial symmetric functions

Consider a set of variables $(x) = x_1, x_2, \ldots, x_d$. A *symmetric* **monomial**

$$\boxed{\boxed{m_\lambda(x) = \sum_\alpha x^\alpha}} \tag{4.1}$$

involves a sum over all distinct permutations α of $(\lambda) = (\lambda_1, \lambda_2, \ldots)$. Thus if $(x) = (x_1, x_2, x_3)$ then

$$m_{21}(x) = x_1^2\, x_2 + x_1^2\, x_3 + x_1\, x_2^2 + x_1\, x_3^2 + x_2^2\, x_3$$
$$m_{1^3}(x) = x_1\, x_2\, x_3$$

The unitary numbering of $(\lambda) = (2\,1)$ with 1, 2, 3 corresponds to the sum of monomials

$$\boxed{\boxed{m_{21}(x) + 2m_{1^3}(x)}}$$

The same linear combination occurs for any number of variables with $d \geq 3$.

The monomials $m_\lambda(x)$ are *symmetric functions*. If $\lambda \vdash n$ then $m_\lambda(x)$ is homogeneous of degree n. Unless otherwise stated we shall henceforth assume that x involves an infinite number of variables x_i.

The *ring of symmetric functions* $\Lambda = \Lambda(x)$ is the vector space spanned by all the $m_\lambda(x)$. This space can be decomposed as

$$\Lambda = \oplus_{n \geq 0} \Lambda^n \tag{4.2}$$

where Λ^n is the space spanned by all m_λ of degree n. Thus the $\{m_\lambda | \lambda \vdash n\}$ form a basis for the space Λ^n which is of dimension $p(n)$ where $p(n)$ is the number of partitions of n. It is of interest to ask if other bases can be constructed for the space Λ^n.

The classical symmetric functions

Three other classical bases are well-known - some since the time of Newton.

1. The elementary symmetric functions
   The $n-$th elementary symmetric function e_n is the sum over all products of n distinct variables x_i, with $e_0 = 1$ and generally

$$e_n = m_{1^n} = \sum_{i_1 < i_2 \ldots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \tag{4.3}$$

   The *generating function* for the e_n is

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} (1 + x_i t) \tag{4.4}$$

2. The complete symmetric functions

The $n-$th complete or *homogeneous* symmetric function h_n is the sum of all monomials of total degree n in the variables x_1, x_2, $\ldots$, with $h_0 = 1$ and generally

$$h_n = \sum_{|\lambda|=n} m_\lambda = \sum_{i_1 \le i_2 \ldots \le i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \tag{4.5}$$

The generating function for the h_n is

$$H(t) = \sum_{n \ge 0} h_n t^n = \prod_{i \ge 1} (1 - x_i t)^{-1} \tag{4.6}$$

3. The power sum symmetric function

The $n-$th power sum symmetric function is

$$p_n = m_n = \sum_{i \ge 1} x_i^n \tag{4.7}$$

The generating function for the p_n is

$$\begin{aligned} P(t) = \sum_{n \ge 1} p_n t^{n-1} &= \sum_{i \ge 1} \sum_{n \ge 1} x_i^n t^{n-1} \\ &= \sum_{i \ge 1} \frac{x_i}{1 - x_i t} \\ &= \sum_{i \ge 1} \frac{d}{dt} \log \frac{1}{1 - x_i t} \end{aligned} \tag{4.8}$$

and hence

$$\begin{aligned} P(t) &= \frac{d}{dt} \log \prod_{i \ge 1} (1 - x_i t)^{-1} \\ &= \frac{d}{dt} \log H(t) \\ &= H^{'}(t)/H(t) \end{aligned} \tag{4.9}$$

Similarly,

$$P(-t) = \frac{d}{dt}\log E(t) = E^{'}(t)/E(t) \tag{4.10}$$

Equation (4.9) leads to the relationship

$$nh_n = \sum_{r=1}^{n} p_r\, h_{n-r} \tag{4.11}$$

It follows from (4.9) that

$$\begin{aligned} H(t) &= \exp\sum_{n\geq 1} p_n\, t^n/n \\ &= \prod_{n\geq 1}\exp(p_n\, t^n/n) \\ &= \prod_{n\geq 1}\sum_{m_n=0}^{\infty}(p_n\, t^n)^{m_n}/n^{m_n}.m_n! \end{aligned} \tag{4.12}$$

and hence

$$H(t) = \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}\, t^{|\lambda|} \tag{4.13}$$

where

$$z_{\lambda} = \prod_{i\geq 1} i^{m_i}.m_i! \tag{4.14}$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i.
Defining

$$\varepsilon_{\lambda} = (-1)^{|\lambda|-\ell(\lambda)} \tag{4.15}$$

we can show in an exactly similar manner to that of Eq.(4.13) that

$$E(t) = \sum_{\lambda} \varepsilon_\lambda z_\lambda^{-1} p_\lambda \, t^{|\lambda|} \tag{4.16}$$

It then follows from Eqs.(4.13) and (4.16) that

$$h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda \tag{4.17}$$

and

$$e_n = \sum_{|\lambda|=n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda \tag{4.18}$$

Exercises

4.1 Show that for $n = 3$

$$\begin{aligned}
p_3 &= x_1^3 + x_2^3 + x_3^3 + \ldots \\
e_3 &= x_1 \, x_2 \, x_3 + x_1 \, x_2 \, x_4 + x_2 \, x_3 \, x_4 + \ldots \\
h_3 &= x_1^3 + x_2^3 + \ldots + x_1^2 \, x_2 + x_1 \, x_2^2 + \ldots + x_1 \, x_2 \, x_3 + x_1 \, x_2 \, x_4 + \ldots
\end{aligned} \tag{4.19}$$

4.2 Noting Eqs. (4.4) and (4.6) and that $H(t)E(-t) = 1$**, show that**

$$\sum_{r=0}^{n} (-1)^r h_{n-r} \, e_r = 0 \tag{4.20}$$

for $n \geq 1$**.**

4.3 Use Eq.(4.20) to show that

$$e_n = det(h_{1-i+j})_{1 \leq i,j \leq n} \tag{4.21}$$

and hence

$$h_n = det(e_{1-i+j})_{1 \leq i,j \leq n} \tag{4.22}$$

4.4 Use Eq.(4.11) to obtain the determinantal expressions

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ ne_n & e_{n-1} & e_{n-2} & \dots & e_1 \end{vmatrix} \tag{4.23}$$

$$n!e_n = \begin{vmatrix} p_1 & 1 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ p_{n-1} & p_{n-2} & . & \dots & n-1 \\ p_n & p_{n-1} & . & \dots & p_1 \end{vmatrix} \tag{4.24}$$

$$(-1)^{n-1}p_n = \begin{vmatrix} h_1 & 1 & 0 & \dots & 0 \\ 2h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ nh_n & h_{n-1} & h_{n-2} & \dots & h_1 \end{vmatrix} \tag{4.25}$$

$$n!h_n = \begin{vmatrix} p_1 & -1 & 0 & \dots & 0 \\ p_2 & p_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ p_{n-1} & p_{n-2} & . & \dots & -n+1 \\ p_n & p_{n-1} & . & \dots & p_1 \end{vmatrix} \tag{4.26}$$

Multiplicative bases for Λ^n

The three types of symmetric functions, h_n, e_n, p_n, do not have enough elements to form a basis for Λ^n, there must be one function for every partition $\lambda \vdash n$. To that end in each case we form *multiplicative* functions f_λ so that for each $\lambda \vdash n$

$$f_\lambda = f_{\lambda_1} f_{\lambda_2} \ldots f_{\lambda_\ell} \tag{4.27}$$

where $f = e$, h, or p Thus, for example,

$$e_{21} = e_2 \cdot e_1 = (x_1 x_2 + x_1 x_3 + x_2 x_3 + \ldots)(x_1 + x_2 + x_3 + \ldots)$$

The Schur functions

The symmetric functions

$$m_\lambda,\ e_\lambda,\ h_\lambda,\ p_\lambda \tag{4.28}$$

where $\lambda \vdash n$ each form a basis for Λ^n. A very important fifth basis is realised in terms of the Schur functions, s_λ, *or for brevity, $S-$functions* which may be variously defined. Combinatorially they may be defined as

$$s_\lambda(x) = \sum_T x^T \tag{4.29}$$

where the summation is over all semistandard $\lambda-$tableaux T. For example, consider the $S-$functions s_λ in just three variables (x_1, x_2, x_3). For $\lambda = (2\,1)$ we have the eight tableaux T found earlier

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \tag{3.8}$$

Each tableaux T corresponds to a monomial x^T to give

$$s_{2\,1}(x_1,\,x_2,\,x_3) = x_1^2\,x_2 + x_1^2\,x^3 + x_1\,x_2^2 + x_1\,x_2\,x_3 + x_1\,x_2\,x_3 + x_1\,x_3^2 + x_2^2\,x_3 + x_2\,x_3^2 \tag{4.30}$$

We note that the monomials in Eq.(4.30) can be expressed in terms of just two *symmetric monomials* **in the three variables** $(x_1,\,x_2,\,x_3)$ **to give**

$$s_{2\,1}(x_1,\,x_2,\,x_3) = m_{2\,1}(x_1,\,x_2,\,x_3) + 2m_{1^3}(x_1,\,x_2,\,x_3) \tag{4.31}$$

In an arbitrary number of variables

$$s_{2\,1}(x) = m_{2\,1}(x) + 2m_{1^3}(x) \tag{4.32}$$

This is an example of the general result that the $S-$function may be expressed as a linear combination of symmetric monomials as indeed would be expected if the $S-$functions are a basis of Λ^n. In fact

$$s_\lambda(x) = \sum_{\mu\vdash n} K_{\lambda\mu} m_\mu \tag{4.33}$$

where $|\lambda| = n$ and $K_{\lambda\lambda} = 1$. The $K_{\lambda\mu}$ are the elements of an upper triangular matrix K known as the Kostka matrix. K is an example of a *transition matrix* **that relates one symmetric function basis to another.**

Calculation of the elements of the Kostka matrix

The elements $K_{\lambda\mu}$ of the Kostka matrix may be readily calculated by the following algorithm :

i. Draw the frame F^λ.

ii. Form all possible semistandard tableaux that arise in numbering F^λ with μ_1 ones, μ_2 twos etc.

iii. $K_{\lambda\mu}$ is the number of semistandard tableaux so formed.

Thus calculating $K_{(42)\,(2^2\,1^2)}$ we obtain the four semistandard tableaux

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&4\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&4\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline 2&3\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline 2&2\\\cline{1-2}\end{array}$$

and hence $K_{(42)\,(2^2\,1^2)} = 4$.

Exercises

4.5 Construct the Kostka matrix for $\lambda, \mu \vdash 4$.

4.6 Show that in the variables (x_1, x_2, x_3) the evaluation of the determinantal ratio

$$\frac{\begin{vmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}}$$

yields the monomial content of the $S-$function s_{21} in three variables as found in Eq.(4.30). N.B. The above exercise is tedious by hand but trivial using MAPLEV.

The last exercise is an example of the classical definition, as opposed to the equivalent combinatorial definition given in Eq.(4.29), given first by Jacobi, namely,

$$s_\lambda = s_\lambda(x_1,\, x_2,\, \ldots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta} \tag{4.34}$$

where λ **is a partition of length** $\leq n$ **and** $\delta = (n-1,\, n-2,\, \ldots,\, 1,\, 0)$ **with**

$$a_{\lambda+\delta} = det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n} \tag{4.35}$$

and

$$a_\delta = \prod_{1\leq i,j\leq n}(x_i - x_j) = det(x_i^{n-j}) \tag{4.36}$$

is the *Vandermonde determinant***.**

Non-standard $S-$**functions**

The $S-$**functions are symmetric functions indexed by ordered partitions** λ**. We shall frequently write** $S-$**functions** $s_\lambda(x)$ **as** $\{\lambda\}(x)$ **or, since we will generally consider the number of variables to be unrestricted, just** $\{\lambda\}$**. As a matter of notation the partitions will normally be written without spacing or commas separating the parts where** $\lambda_i \leq 9$**. A space will be left after any part** $\lambda_i \geq 10$**. Thus we write** $\{12, 11, 9, 8, 3, 2, 1\} \equiv \{12\ \ 11\ \ 98321\}$ **While we have defined the** $S-$**function in terms of ordered partitions we sometimes encounter** $S-$**functions that are not in the standard form and must convert such** *non-standard* $S-$**functions into standard** $S-$**functions. Inspection of the determinantal**

forms of the $S-$function leads to the establishment of the following *modification rules* :

$$\{\lambda_1, \lambda_2, \ldots, -\lambda_\ell\} = 0 \tag{4.37}$$

$$\{\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_\ell\} = -\{\lambda_1, \ldots, \lambda_{i+1} - 1, \lambda_i + 1, \ldots, \lambda_\ell\} \tag{4.38}$$

$$\{\lambda\} = 0 \qquad \textbf{if } \lambda_{i+1} = \lambda_i + 1 \tag{4.39}$$

Repeated application of the above three rules will reduce any non-standard $S-$function to either zero or to a signed standard $S-$function. In the process of using the above rules trailing zero parts are omitted

---

Exercise
4.7 Show that

$$\{24\} = -\{3^2\}, \quad \{141\} = -\{321\}$$
$$\{14-25-14\} = -\{3^3 2\}$$
$$\{3042\} = 0, \quad \{3043\} = \{3^2 2\}$$

Skew $S-$functions

The combinatorial definition given for $S-$functions in Eq.(4.29) is equally valid for skew tableaux and can hence be used to define *skew* $S-$functions $s_{\lambda/\mu}(x)$ or $\{\lambda/\mu\}$. Since the $s_{\lambda/\mu}(x)$ are symmetric functions they must be expressible in terms of $S-$functions $s_\nu(x)$ such that

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu \tag{4.40}$$

It may be shown that the coefficients $c^{\lambda}_{\mu\nu}$ are necessarily non-negative integers and symmetric with respect to μ and ν. The coefficients $c^{\lambda}_{\mu\nu}$ are commonly referred to as *Littlewood-Richardson* **coefficients.**

The Littlewood-Richardson rule

The product of two $S-$functions can be written as a sum of $S-$functions, viz.

$$s_\mu . s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda \tag{4.41}$$

The Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ in Eqs. (4.40) and (4.41) are identical, though the summations are of course different. In both cases $|\mu|+|\nu| = |\lambda|$. A rule for evaluating the coefficients $c^\lambda_{\mu\nu}$ was given by Littlewood and Richardson in 1934 and has played a major role in all subsequent developments. The rule may be stated in various ways. We shall state it first in terms of semistandard tableaux and then also give the rule for evaluating the product given in Eq.(4.41) which is commonly referred to as the *outer multiplication* **of $S-$functions. In each statement the concepts of a** *row-word* **and of a** *lattice permutation* **is used.**

Definition 4.1 A word

Let T be a tableau. From T we derive a row-word or sequence $w(T)$ by reading the symbols in T from right to left (i.e. as in Arabic or Hebrew) in successive rows starting at the top row and proceeding to the bottom row

Thus for the tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & 2 & 2 & 3 \\
\hline
2 & 2 & 3 & 3 \\
\cline{1-4}
4 & 4 \\
\cline{1-2}
5 & 6 \\
\cline{1-2}
7 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}$$

we have the word $w(T) = 322113322446578$ and for the skew tableau

$$\begin{array}{cccc} & & \boxed{1} & \boxed{1}\ \boxed{1} \\ & \boxed{1} & \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{2} & & \end{array}$$

we have the word $w(T) = 11122121$.

Definition 4.2 A lattice permutation

> *A word $w = a_1 a_2 \ldots a_N$ in the symbols $1, 2, \ldots, n$ is said to be a lattice permutation if for $1 \leq r \leq N$ and $1 \leq i \leq n-1$, the number of occurrences of the symbol i in $a_1 a_2 \ldots a_r$ is not less than the number of occurrences of $i + 1$.*

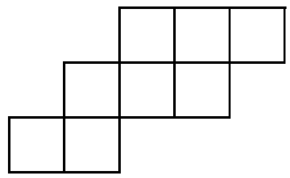
Thus the word $w(T) = 322113322446578$ is clearly not a lattice permutation whereas the word $w(T) = 11122121$ is a lattice permutation. The word $w(T) = 12122111$ is not a lattice permutation since the sub-word 12122 has more twos than ones.

Theorem 4.1 *The value of the coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of semistandard tableaux T of shape $F^{\lambda/\mu}$ and content ν such that $w(T)$ is a lattice permutation.*

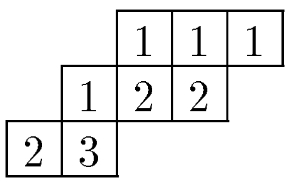
By *content ν* we mean that each tableau T contains ν_1 ones, ν_2 twos, etc.

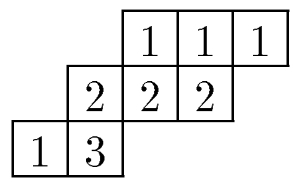
Example

Let us evaluate the coefficient $c^{\{542\}}_{\{431\}\{21\}}$. We first draw the frame $F^{\{542/21\}}$.

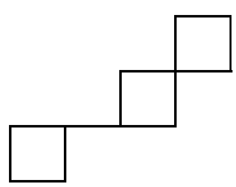


Into this frame we must inject the content of $\{431\}$ i.e. 4 ones, 3 twos and 1 three in such a way that we have a lattice permutation. We find two such numberings

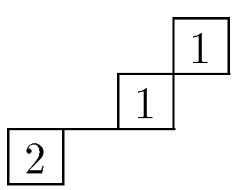


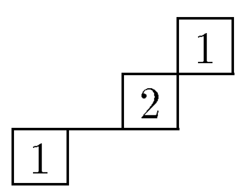


and hence $c^{\{542\}}_{\{431\}\{21\}} = 2$. Note that in the evaluation we had a choice, we could have, and indeed more simply, evaluated $c^{\{542\}}_{\{21\}\{431\}}$. In that case we would have drawn the frame $F^{\{542/431\}}$ to get



Note that in this case the three boxes are disjoint. This skew frame is to be numbered with two ones and one 2 leading to the two tableaux





verifying the previous result. Theorem 4.1 gives a direct method for evaluating the Littlewood-Richardson coefficients. These coefficients can be used to evaluate both skews and products. It is sometimes useful to state a procedure for directly evaluating products.

Theorem 4.2 *to evaluate the $S-$function product $\{\mu\}.\{\nu\}$*

1. *Draw the frame F^{μ} and place ν_1 ones in the first row, ν_2 twos in the second row etc until the frame is filled with integers.*
2. *Draw the frame F^{ν} and inject positive integers to form a semistandard tableau such that the word formed by reading from right to left starting at the top row of the first frame and moving downwards along successive rows to the bottom row and then continuing through the second frame is a lattice permutation.*
3. *Repeat the above process until no further words can be constructed.*
4. *Each word corresponds to an $S-$function $\{\lambda\}$ where λ_1 is the number of ones, λ_2 the number of twos etc.*

As an example consider the $S-$function product $\{21\}.\{21\}$.

Step 1 gives the tableau

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Steps 2 and 3 lead to the eight numbered frames

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}$$

Step 4 then lead to the eight words

$$\begin{array}{cccc} 112112 & 112113 & 112212 & 112213 \\ 112312 & 112314 & 112323 & 112324 \end{array}$$

from which we conclude that

$$\{21\}.\{21\} = \{42\}+\{41^2\}+\{3^2\}+2\{321\}+\{31^3\}+\{2^3\}+\{2^21^2\}$$

Exercises
4.8 Show that $c^{\{75321^3\}}_{\{4321\}.\{4321\}} = 8$.
4.9 Show that

$$\begin{aligned}\{31\}.\{31\} =&\{62\} + \{61^2\} + \{53\} + 2\{521\} + \{51^3\} + \{4^2\} \\ &+ 2\{431\} + \{42^2\} + \{421^2\} + \{3^22\} + \{3^21^2\}\end{aligned}$$

4.10 Show that

$$\{321/21\} = \{3\} + 2\{21\} + \{1^3\}$$

Relationship to the unitary group

We have explored various symmetric functions indexed by partitions and defined on sets of variables. The variables can admit many interpretations. In some instances we may choose a set of variables $1, q, q^2, \ldots, q^n$ or we could even use a set of matrices. The link between $S-$functions and the character theory of groups is such that, if λ is a partition with $\ell(\lambda) \leq N$ and the

eigenvalues of a group element, g, of the unitary group U_N are given by $x_j = \exp(i\phi_j)$ for $j = 1, 2, \ldots, N$ then the $S-$function

$$\{\lambda\} = \{\lambda_1\lambda_2\ldots\lambda_N\} = s_\lambda(x)$$
$$= s_\lambda(\exp(i\phi_1)\exp(i\phi_2)\ldots\exp(i\phi_N))$$

is nothing other than the character of g in the irreducible representation of U_N conventionally designated by $\{\lambda\}$.

The Littlewood-Richardson rule gives the resolution of the Kronecker product $\{\mu\} \times \{\nu\}$ of U_N as

$$\{\mu\} \times \{\nu\} = \sum_{|\lambda|=|\mu|+|\nu|} c_{\{\mu\}.\{\nu\}}^{\{\lambda\}}\{\lambda\} \tag{4.42}$$

where the $c_{\{\mu\}.\{\nu\}}^{\{\lambda\}}$ are the usual Littlewood-Richardson coefficients. Equation (4.42) must be modified for partitions λ involving more than N parts. Here the *modification rule* is very simple. We simply discard all partitions involving more than N parts. We shall return to these matters later in this course when we use our results to discuss the classification of many-electron states, especially for the electronic $f-$shell.

Symmetry and Spectroscopic Calculations

Lecture Five

5.1 $S-$function series

Infinite series of $S-$functions play an important role in determining branching rules and furthermore lead

to concise symbolic methods well adapted to computer implementation. Consider the infinite series

$$\begin{aligned} L &= \prod_{i=1}^{\infty}(1 - x_i) \\ &= 1 - \sum x_1 + \sum x_1 x_2 - \dots \end{aligned} \tag{5.1}$$

where the summations are over all distinct terms. e.g.

$$\sum x_1 x_2 = x_1 x_2 + x_1 x_3 + \ldots + x_2 x_3 + x_2 x_4 + \ldots \tag{5.2}$$

Recalling Eq.(4.3) we see that Eq.(5.1) is simply a signed sum over an infinite set of elementary symmetric functions e_n with

$$e_n = m_{1^n} = s_{1^n} = \{1^n\} \tag{5.3}$$

and hence Eq.(5.1) may be written as an infinite sum of $S-$functions such that

$$\begin{aligned} L &= 1 - \{1\} + \{1^2\} - \{1^3\} + \dots \\ &= \sum_{m=0}^{\infty}(-1)^m \{1^m\} \end{aligned} \tag{5.4}$$

We may define a further infinite series of $S-$functions by taking the inverse of Eq.(5.1) to get

$$\begin{aligned} M &= \sum_{i=1}^{\infty}(1 - x_i)^{-1} \\ &= 1 + \{1\} + \{2\} + \dots \\ &= \sum_{m=0}^{\infty}\{m\} \end{aligned} \tag{5.5}$$

Clearly

$$LM = 1 \tag{5.6}$$

a result that is by no means obvious by simply looking at the product of the two series.

In practice large numbers of infinite series and their associated generating functions may be constructed. We list a few of them below:

$$\boxed{\begin{array}{ll} \mathbf{A} = \sum_\alpha (-1)^{w_\alpha}\{\alpha\} & \mathbf{B} = \sum_\beta \{\beta\} \\ \mathbf{C} = \sum_\gamma (-1)^{w_\gamma/2}\{\gamma\} & \mathbf{D} = \sum_\delta \{\delta\} \\ \mathbf{E} = \sum_\epsilon (-1)^{(w_\epsilon+r)/2}\{\epsilon\} & \mathbf{F} = \sum_\zeta \{\zeta\} \\ \mathbf{G} = \sum_\epsilon (-1)^{(w_\epsilon-r)/2}\{\epsilon\} & \mathbf{H} = \sum_\zeta (-1)^{w_\zeta}\{\zeta\} \\ \mathbf{L} = \sum_m (-1)^m\{1^m\} & \mathbf{M} = \sum_m \{m\} \\ \mathbf{P} = \sum_m (-1)^m\{m\} & \mathbf{Q} = \sum_m \{1^m\} \end{array}} \tag{5.7}$$

where (α) and (γ) are mutually conjugate partitions, which in the Frobenius notation take the form

$$(\alpha) = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1+1 & a_2+1 & \dots & a_r+1 \end{pmatrix} \tag{5.8a}$$

and

$$(\gamma) = \begin{pmatrix} a_1+1 & a_2+1 & \dots & a_r+1 \\ a_1 & a_2 & \dots & a_r \end{pmatrix} \tag{5.8b}$$

(δ) is a partition into *even parts* only and (β) is conjugate to (δ). (ζ) is any partition and (ϵ) is any self-conjugate partition. r is the Frobenius rank of (α), (γ) and (ϵ).

These series occur in mutually inverse pairs:

$$AB = CD = EF = GH = LM = PQ = \{0\} = 1 \tag{5.9}$$

Furthermore,

$$\begin{aligned} LA = PC = E \quad & MB = QD = F \\ MC = AQ = G \quad & LD = PB = H \end{aligned} \tag{5.10}$$

We also note the series

$$R = \{0\} - 2\sum_{a,b}(-1)^{a+b+1}\begin{pmatrix} a \\ b \end{pmatrix} \qquad S = \{0\} + 2\sum_{a,b}\begin{pmatrix} a \\ b \end{pmatrix} \tag{5.11}$$

where we have again used the Frobenius notation, and

$$\begin{aligned} V = \sum_{\omega}(-1)^q\{\tilde{\omega}\} \quad & W = \sum_{\omega}(-1)^q\{\omega\} \\ X = \sum_{\omega}\{\tilde{\omega}\} \quad & Y = \sum_{\omega}\{\omega\} \end{aligned} \tag{5.12}$$

where (ω) is a partition of an even number into at most two parts, the second of which is q, and $\tilde{\omega}$ is the conjugate of ω. We have the further relations

$$RS = VW = \{0\} = 1 \tag{5.13}$$

and

$$\begin{aligned} PM = AD = W \quad & LQ = BC = V \\ MQ = FG = S \quad & LP = HE = R \end{aligned} \tag{5.14}$$

5.2 Symbolic manipulation

The above relations lead to a method of describing many of the properties of groups via symbolic manipulation of infinite series of $S-$functions. Thus if $\{\lambda\}$

is an $S-$function then we may symbolically write, for example,

$$\{\lambda/M\} = \sum_m \{\lambda/m\} \tag{5.15}$$

We can construct quite remarkable identities such as:

$$BD = \sum_\zeta \{\zeta\} \cdot \{\zeta\} \tag{5.16}$$

or for an arbitrary $S-$function $\{\epsilon\}$

$$BD \cdot \{\epsilon\} = \sum_\zeta \{\zeta\} \cdot \{\zeta/\epsilon\} \tag{5.17}$$

Equally remarkably we can find identities such as

$$\{\sigma\cdot\tau\}/Z = \{\sigma/Z\}\cdot\{\tau/Z\} \quad \textbf{for} \quad Z = L, M, P, Q, R, S, V, W \tag{5.18a}$$

$$\{\sigma\cdot\tau\}/Z = \sum_\zeta \{\sigma/\zeta Z\}\cdot\{\tau/\zeta Z\} \quad \textbf{for} \quad Z = B, D, F, H \tag{5.18b}$$

$$\{\sigma\cdot\tau\}/Z = \sum_\zeta (-1)^{w_\zeta}\{\sigma/\zeta Z\}\cdot\{\tau/\tilde{\zeta} Z\} \quad \textbf{for} \quad Z = A, C, E, G \tag{5.18c}$$

These various identities can lead to a symbolic method of treating properties of groups particulary amenable to computer implementation.

5.3 The $U_n \to U_{n-1}$ branching rule

As an illustration of the preceding remarks we apply the properties of $S-$functions to the determination of

the $U_n \to U_{n-1}$ branching rules. The vector irrep $\{1\}$ of U_n can be taken as decomposing under $U_n \to U_{n-1}$ as

$$\{1\} \to \{1\} + \{0\} \tag{5.19}$$

that is into a vector $\{1\}$ and scalar $\{0\}$ of U_{n-1}. In general, the spaces corresponding to tensors for which a particular number of indices, say m, take on the value n, define invariant subspaces. Such indices must be mutually symmetrised. The irreducible representations specified by the quotient $\{\lambda/m\}$ are those corresponding to tensors obtained by contracting the indices of the tensor corresponding to $\{\lambda\}$ with an $m-$th rank symmetric tensor. Thus we may symbolically write the general branching rule as simply

$$\{\lambda\} \to \{\lambda/M\} \tag{5.20}$$

Thus for example under $U_3 \to U_2$ we have

$$\begin{aligned} \{21\} &\to \{21/M\} \\ &\to \{21/0\} + \{21/1\} + \{21/2\} \\ &\to \{21\} + \{2\} + \{11\} + \{1\} \end{aligned} \tag{5.21}$$

5.4 The Gel'fand states and the betweenness condition

The so-called Gel'fand states play an important role in the Unitary Group Approach (UGA) to many-electron theory. This comes about from considering the canonical chain of groups

$$U_n \supset U_{n-1} \supset \ldots U_2 \supset U_1 \tag{5.22}$$

The states of such a chain follow directly from consideration of Eq.(5.20). Each state may be represented by a triangular array having n rows. There are n entries $m_{i,n}$ with $i = 1, 2, \ldots, n$ corresponding to the usual partition (λ) padded out with zeroes to fill the row if need be. The second row contains $n-1$ entries $m_{i,n-1}$ placed below the first row so that the entry $m_{1,n-1}$ occurs between the entries $m_{1,n}$ and $m_{2,n}$ etc. Each successive row contains one less entry with the bottom row containing just one entry $m_{1,1}$. The number of such states is just the dimension of the irrep $\{\lambda\}$ of U_n.
Consider the irrep of U_3 labelled as $\{21\}$. We find the eight Gel'fand states

$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{pmatrix} \qquad \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{pmatrix}$$
$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{pmatrix} \qquad \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix}$$
$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{pmatrix} \qquad \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix}$$
$$\begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{pmatrix} \qquad \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{pmatrix}$$

5.5 Using SCHUR to evaluate properties of $S-$functions

All of the properties of $S-$functions we have so far

discussed, and many more, can be readily found using the programme SCHUR which has been placed on on the Pc's here for your experimentation.It will do many things besides just $S-$functions. e.g. Properties of irreps of all the compact Lie groups such as dimensions, Kronecker products and branching rules. However at this stage we will restrict our attention to $S-$functions. Later on in this course we will look at other features. The programme as installed has a principal file SCHUR.EXE and a large number of HELP files. Ignore all other files at this stage. Go to the SCHUR directory and enter the command SCHUR and after a few moments your screen should look like

```
SCHUR          #0333
User:Students
Site:Instytut Fizyki
Uniwersytet Mikolaja Kopernika
ul.  Grudziadzka 5/7
87-100 Torun
POLAND
Copyright.  Distribution and copying prohibited
[Version 5.0] (c) Schur Software Associates 1984,1986,1987
,1988,1989
(If you wish to EXIT, enter 'END')
(If you wish to obtain HELP, enter ?'help')
DPrep Mode (with function)
DP>
-
```

Note that you can EXIT the programme any time by entering END. Also while I will indicate commands to be entered in CAPITALS the entry of commands

is not case sensitive. To get to the $S-$function mode enter SFN and you will see

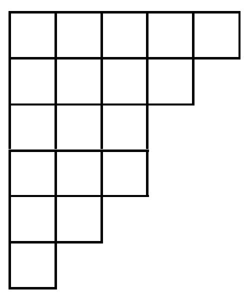
```
DP>
SFN
Schur Function Mode
SFN>
```

- You can obtain a list of commands in the Schur Function Mode by entering ?'SFNMODE' to give

```
SFN>
?'SFNMODE'
SFNmode
This mode does all calculations involving Sfns.
Commands available are:
ABsval, ADd, ALARM, ALlskew, ATtach, BELl,CDiv ,CLEave,
COeffs,COLour,CONJ, CUT, DEtach,DIGits,DIR,DISK,DIStinct,
DPMode, END, EXit, FACtor, FN,FRame, FULL, FULLSA,
FSA,
HALlp, Inner, INSert, LAPs, LAst, LEngth,LIMit, LOad,
LOG, LRAIse, MCount, MKWeight, MORe, MUlt, NLIMit,
Outer,
PAUSE, PHase, PLeth, PLInner, POwer, QEXpand, QFN,
QOUTer,
QQExpand, QQSeries, QSEries, QSKew, QSTD, RAIse,
RCOnvert,
REDuce, REM, REPmode, REVerse, RInner, RQINner, RL-
RAIse,
RRaise, SAMewt, SAve, SCONvert, SETSfn, SKew, SQIN-
ner,
STAtus,STD,STIme, SUb,SVar,TCount,TIme, TRunc,
TRWt, WEight, Zero.
```

```
    Some of the Sfn commands make use  of the Sfn in-
finite
    series (SKew,TRunc, TRWt).
    The Sfn series in Schur are:
    A, B, C, D, E, F, G, H, L, M, P, Q, R, S, T, V, W,
X, Y.
    These series may be accessed by upper or lower case
    letters.
    SFN>
    -
```

Many of the commands you won't need to consider at the beginning. Each command has it's own helpfile. Try entering the command FRAME 54321 and you should see on your screen the frame F^{54321} drawn as



The following give examples of syntax as explained in the Helpfile DIGITS and SYNTAX. Try other frames such as 5322211 which could be entered as either FRAME532$\hat{3}$1$\hat{2}$ or as just FRAME5322211 or even as FRA5322211. To draw the frame for the partition 12 10 43221 you enter FRA !12 !10 4321 . Note that the exclamation mark (!) is put in front of digits larger than 9 and a space then follows the digits. Spaces are optional for numbers $<= 9$. If you enter FRA5.4321 you will see on the screen the frame F^{4321} with the

digit 5 above it. To see the significance of that try entering OUTER 21,21 and you will obtain the output

```
SFN>
OUTER 21,21
{42} + {41^2 } + {3^2 } + 2{321} + {31^3 }
+ {2^3 } + {2^2 1^2 }
SFN>
```

- Notice that the $S-$function $\{321\}$ appears with a multiplicity of 2. Now enter FRAME LAST and you will see the frames for each partition drawn on the screen with a 2 appearing above the frame for $\{321\}$. Now try the command FRAME OUTER 21,21 and you will start to learn how you can combine sequences of commands. Enter OUTER 4321,4321 and note that you get a screen full of $S-$functions with the word MORE appearing on the left. Pressing a key will show you the next screen full. You can turn off MORE by entering MORE FALSE now repeat OUTER 4321,4321 and 206 $S-$functions will scroll by. Try FRAME OUTER 4321,4321 and 206 frames will flash by with their associated multiplicities. To count the number of frames simply enter TCOUNT LAST and to count the sum of the multiplicities enter MCOUNT LAST.

Look at the helpfiles associated with the commands SKEW, TRWT, WEIGHT, LENGTH and see if you can determine the terms in each of the $S-$function series we have discussed today up to say weight 8. Feel free to explore the various features. See if you can make yourself a LOGFILE in which you obtain the aforementioned series and then edit the logfile and

print out a neat table with suitable captions etc., possibly as a TEXfile.

Some Relevant Literature

The following references have been chosen to give a general background to the literature relevant to this course. I shall at various times refer to them by number.

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