Symmetric Functions and the Symmetric Group 9
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Mathematics requires a small dose, not of genius, but of imaginative freedom which, in a larger dose, would be insanity. And if mathematicians tend to burn out early in their careers, it is probably because life has forced them to acquire too much common sense, thereby rendering them too sane to work. But by then they are sane enough to teach, so a use can still be found for them.
— Angus K Rodgers

■ 9.1 Quantum Dots and Symmetry Physics

The subject of quantum dots involves the confinement of N electrons in two or three dimensions, commonly by electrostatic fields, over a nano-metre scale. The confining potential is, to a good approximation parabolic. The quantum dot behaves as an $N-$electron atom without a nuclear core. One may add or subtract a single electron from a quantum dot giving rise to the possibility of nano-metre scale devices such as transitors etc.

In an atom the kinetic energy tends to dominate over the potential energy (the confinement length is small) whereas in a quantum dot the two contributions are roughly of the same order making normal perturbative methods difficult. A closely analogous problem is that of nucleons confined in a harmonic oscillator potential with quantised motion occuring about the centre of mass of the $N-$nucleon system. We shall first review some of the properties of the isotropic harmonic oscillator, the unitary group $U(3)$ and the special unitary group $SU(3)$.

■ 9.2 The Isotropic harmonic oscillator

The Hamiltonian H of a normalised isotropic harmonic oscillator (i.e. with $m = \hbar = \omega = 1$) in three-dimensions may be written as

$$H = \frac{1}{2}(\mathbf{p}^2 + \mathbf{r}^2) \tag{9.1}$$

From Heisenberg's quantisation postulate the coordinates q_i and momenta p_i satisfy the commutation relations

$$[q_i, q_j] = [p_i, p_j] = 0, \qquad [q_i, p_j] = i\delta_{ij} \tag{9.2}$$

Now introduce boson annihilation and creation operators (a and $\mathrm{a}^\dagger$ respectively)

$$\mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{r} + i\mathbf{p}), \qquad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}(\mathbf{r} - i\mathbf{p}) \tag{9.3}$$

which satisfy the bosonic commutation relation

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{9.4}$$

The Hamiltonian can now be written as

$$H = \mathbf{a}^\dagger \cdot \mathbf{a} + \frac{3}{2} \tag{9.5}$$

Use of Eqn. (9.4) then leads to

$$[H, a_j^\dagger] = a_j^\dagger, \qquad [H, a_j] = -a_j \tag{9.6}$$

Thus we deduce that $a_j^\dagger$ creates and a_j annihilates a quantum in the j direction. We recognise $\mathbf{a}^\dagger \cdot \mathbf{a}$ as being the *number operator* with eigenvalues of

$$n = n_1 + n_2 + n_3 \tag{9.7}$$

and hence the energy eigenvalues of H are

$$E_n = n + \frac{3}{2} \quad (n = 0, 1, 2, \ldots) \tag{9.8}$$

with normalised state vectors

$$|n_1n_2n_3\rangle = \prod_{i=1}^{3} \frac{a_i^{\dagger n_i}}{\sqrt{n_i!}}|000\rangle \tag{9.9}$$

with $|000\rangle$ being the vacuum state with

$$a_j|000\rangle = 0 \tag{9.10}$$

Noting that $\mathrm{a}^\dagger = \mathrm{a}^*$ we have

$$\langle n_1n_2n_3| = \langle 000| \prod_{i=1}^{3} \frac{a_i^{n_i}}{\sqrt{n_i!}} \tag{9.11}$$

with

$$\langle 000|a_j^\dagger = 0 \tag{9.12}$$

■ 9.3 Degeneracy Group of the Isotropic Harmonic Oscillator

Let us introduce nine operators

$$T_{ij} = \frac{1}{2}\{a_i^\dagger, a_j\} \qquad (i,j = 1,2,3) \tag{9.13}$$

where $\{a,b\} \equiv ab + ba$. Using the basic boson commutation relations of Eqn. (9.4) we find

$$[T_{ij}, T_{rs}] = \delta_{jr}T_{is} - \delta_{is}T_{rj} \tag{9.14}$$

Thus the nine operators T_{ij} close under commutation and generate a Lie algebra. Putting $H_i \equiv T_{ii}$ (do not confuse this with the Hamiltonian) we find the three H_i form a self-commuting set and

$$[H_i, T_{jr}] = (\delta_{ij} - \delta_{ir})T_{jr} \tag{9.15}$$

all the roots are of the form $e_i - e_j$ where the e are mutually orthogonal unit vectors.

The set of nine operators T_{ij} may be identified as the generators of the unitary group in three dimensions, $U(3)$. The Hamiltonian H is related to the H_i of Eqn. (9.15) via

$$H = H_1 + H_2 + H_3 \tag{9.16}$$

commutes with all T_{ij}. The three operators

$$H' = H_i - \frac{H}{3} \tag{9.17}$$

taken with the $T_{ij} \quad (i \neq j)$ can be taken as the generators of the special unitary group $SU(3)$ if we remember that since $\sum_i H_i' = 0$ the H_i' are not linearly independent. For reasons that will become apparent shortly we refer to $U(3)$ as the degeneracy group of the isotropic harmonic oscillator.

■ 9.4 Labelling Representations and Weights

In the case of the angular momentum group $SO(3)$ we label the angular momentum states as $|JM\rangle$ where M is the eigenvalue of J_z with J being the *highest weight* of M. This idea carries over to Lie groups in general. We recall that in the case of $SO(3)$ we can write the defining commutation relations as

$$[J_z, J_\pm] = \pm L_\pm \qquad [J_+, J_-] = J_z \tag{9.17}$$

with

$$J_\pm = \frac{1}{\sqrt(2)}(J_x \pm iJ_y) \tag{9.18}$$

For a general semisimple Lie algebra of rank ℓ we have ℓ operators ,$H_i \quad (i = 1, \ldots, \ell)$, that commute among themselves. The Lie algebra can be cast into the standard Cartan-Weyl form as

$$\begin{aligned}
[H_i, H_j] &= 0 \qquad (i,j = 1, \ldots \ell) \\
[H_i, E_\alpha] &= \alpha_i E_\alpha \\
[E_\alpha, E_\beta] &= N_{\alpha\beta}E_{\alpha+\beta} \\
[E_\alpha, E_{-\alpha}] &= \alpha^i H_i
\end{aligned} \tag{9.19}$$

where the E_α are the analogues of the ladder operators $J_\pm$ of SO_3.

Just as in SO_3 we distinguish the components of a representation by the eigenvalues of J_z for a Lie group we may label the components of a representation by the eigenvalues of the ℓ self-commuting operators H_i. For any compact Lie algebra the *highest* weight vector is unique and hence can be used to specify the representation. Consider for example, the group $U(3)$ which has three self-commuting operators H_i. Suppose we wish to determine the representation of $U(3)$ whose components are the annihilation a and creation operators $a^\dagger$, we have

$$[H_i, a_j^\dagger] = \delta_{ij} a_j^\dagger \quad \text{and} \quad [H_i, a_j] = -\delta_{ij} a_j \tag{9.20}$$

Thus the components of $\mathbf{a}^\dagger$ give rise to the set of weight vectors $(100), (010), (001)$. The highest weight vector is (100) and hence we can label the representation as $\{100\}$ of $U(3)$. Likewise, the components of a give rise to the weight vectors $(-100), (0-10), (00-1)$. We say that a weight vector w is higher than a weight vector w' if the first component of their difference $w - w'$ is *positive.* Thus the highest weight for a is $(00-1)$ and the representation of $U(3)$ spanned by the components of a may be labelled as $\{00-1\}$ which is *contragredient* to $\{100\}$.

■ Exercises

9.1 Noting Eqn(9.14) show that the nine operators T_{ij} are associated with the nine weight vectors (000), (000), (000), (1-10), (10-1), (01-1), (-110), (-101), (0-11).

9.2 Determine the highest weight vector in the above set of weight vectors.

9.3 Repeat the above analysis for a two-dimensional isotropic harmonic oscillator and show that the relevant symmetry group is $U(2)$.

■ 9.5 Rotational Symmetry and the Isotropic Harmonic Oscillator

The harmonic oscillator Hamiltonian, Eqn. (9.1), commutes with all the components of the angular momentum operator

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = i\mathbf{a} \times \mathbf{a}^\dagger \tag{9.21}$$

and hence H is rotationally invariant. The components of $\mathbf{L}$ form under commutation the Lie algebra associated with the group $SO(3)$. Noting the definition of the operators T_{ij}, Eqn.(9.13), and Eqn. (9.21) we have

$$L_1 = -i(T_{23} - T_{32}), \quad L_2 = -i(T_{31} - T_{13}), \quad L_3 = -i(T_{12} - T_{21}) \tag{9.22}$$

We may choose L_3 as the generator of the group $SO(2)$ and hence for the three-dimensional isotropic harmonic oscillator we have the group structure

$$U(3) \supset SU(3) \supset SO(3) \supset SO(2) \tag{9.23}$$

It is convenient to label the oscillator states in a basis $|n\ell m\rangle$ where $n = 0, 1, 2, \ldots$. We have

$$n = 2x + \ell \qquad \text{with} \quad x = 0, 1, 2, \ldots \tag{9.23}$$

and hence the values of ℓ associated with a given value of n are

$$\begin{aligned} \ell &= 1, 3, 5, \ldots, n \qquad n \quad \textit{odd} \\ &= 0, 2, 4, \ldots, n \qquad n \quad \textit{even} \end{aligned} \tag{9.24}$$

and thus for a given n there is a set of $\frac{(n+1)(n+2)}{2}$-fold degenerate states $|n\ell m\rangle$. This is precisely the dimension of the symmetric representation of $U(3)$ designated by the partition $\{n, 0, 0\}$ and hence the statement that the group $U(3)$ is the *degeneracy group* of the three-dimensional isotropic harmonic oscillator.

$n = 5$	p, f, h
$n = 4$	s, d, g
$n = 3$	p, f
$n = 2$	s, d
$n = 1$	p
$n = 0$	s

The first six levels of the isotropic harmonic oscillator

In the preceding we have developed the theory for a *single* particle in a harmonic oscillator potential. This particle could equally well be a nucleon as in nuclear physics or an electron in a quantum dot. The degeneracies are exactly the same as is the form of the energy spectrum. To proceed further requires we develop a many-particle model for particles interacting in a harmonic oscillator potential. To that end we may seek to develop a *dynamical group*.

Two combinatorial observations

These notes are supplementary to Symmetric Functions 9 and concern

1. Some additional remarks on boson-fermion symmetry.
2. An observation relating to the Littlewood-Richardson Rule.

1. Boson-Fermion symmetry for a one-dimensional harmonic oscillator

The $n-$ dimensional isotropic harmonic oscillator has the metaplectic group $Mp(2n)$ as its dynamical group with $U(n)$ as the degeneracy group. The complete set of states span the infinite dimensional unitary irreducible representation $\tilde{\Delta}$ of $Mp(2n)$. Under $Mp(2n) \to U(n)$ one has the branching rule

$$\tilde{\Delta} \to M \tag{1}$$

where

$$M = \sum_{m=0}^{\infty} \{m\} \tag{2}$$

Consider N non-interacting particles in an $n-$dimensional harmonic oscillator potential. In general these particles will form states belonging to symmetrised powers (or plethysms) according to the various partitions of the integer N i.e. with respect to the group $U(n)$ terms coming from the plethysm

$$M \otimes \{\lambda\} \tag{3}$$

Now consider the special case of $n = 1$ with either N bosons or fermions. The degeneracy group is now just $U(1)$ and we can readily evaluate Eq.(3) for the totally symmetric and totally antisymmetric cases as plethysms. At the $U(1)$ level we have for the $M-$series

$$M \otimes \{N\} = \sum_k g_N^k \{k\} \tag{4}$$

where g_N^k is the number of partitions of k into at most N parts with repetitions and null parts allowed and

$$M \otimes \{1^N\} = \sum_\ell c_N^\ell \{\ell\} \tag{5}$$

where c_N^ℓ is the number of partitions of ℓ into N distinct parts, including the null part.

If $\ell = k + (N^2 - N)/2$ then we have the identity

$$c_N^\ell = g_N^k \tag{6}$$

For example,

$$M \otimes \{4\} \supset \{0\} + \{1\} + 2\{2\} + 3\{3\} + 5\{4\} + 6\{5\} + 9\{6\} + 11\{7\} + 15\{8\} + 18\{9\} \tag{7}$$

$$M \otimes \{1^4\} \supset \{6\} + \{7\} + 2\{8\} + 3\{9\} + 5\{10\} + 6\{11\} + 9\{12\} + 11\{13\} + 15\{14\} + 18\{15\} \tag{8}$$

$$c_N^{13} = g_N^7$$

For g_4^7 and c_4^{13} we have the respective sets of 11 partitions

$$g_4^7\{2^31\} + \{321^2\} + \{32^2\} + \{3^21\} + \{41^3\} + \{421\} + \{43\} + \{51^2\} + \{52\} + \{61\} + \{7\}$$

$$c_4^{13}\{5431\} + \{6421\} + \{643\} + \{652\} + \{7321\} + \{742\} + \{751\} + \{832\} + \{841\} + \{931\} + \{1021\}$$

The identity, Eq.(6), comes about by realising that one can map from one of the sets of partitions to the other by adding or subtracting $\rho_N = (N-1, \ldots, 2, 1, 0)$. Adding ρ to the partitions of k into at most N parts, converts them into partitions, all of whose parts are distinct. Hence $c^\ell = g^k if \ell = k + \frac{1}{2}N(N-1)$. Thus in the example above add $(3, 2, 1, 0)$ to the g_4^7 list gives that of c_4^{13}.

The consequence of the boson-fermion equivalence is that the thermodynamic properties of $N-$non-interacting bosons or fermions are essentially equivalent apart from a shift in the groundstate.

2. Littlewood-Richardson coefficients

Kirillov has noted that if $c^{\lambda}_{\mu\nu} = 1$ then

$$c^{N\lambda}_{N\mu,N\nu} = 1 \tag{1}$$

His observation can be conjectured to generalise to

$$c^{N\lambda}_{N\mu,N\nu} = \binom{N+k-1}{k-1} \quad if \quad c^{\lambda}_{\mu\nu} = k \tag{2}$$

where in both cases N multiplies all the parts of the attached partition. e.g.

$$\{321\} \cdot \{431\} \supset$$
$$4\{24\ 20\ 84\} + 4\{24\ 16\ 12\ 4\} + 4\{24\ 16\ 84^2\} + 4\{20\ 16\ 12\ 8\} + 4\{20\ 16\ 12\ 4^2\} + 4\{20\ 16\ 8^24\} \tag{3}$$

and

$$\{12\ 84\} \cdot \{16\ 12\ 4\} \supset$$
$$35\{24\ 20\ 84\} + 35\{24\ 16\ 12\ 4\} + 35\{24\ 16\ 84^2\} + 35\{20\ 16\ 12\ 8\} + 35\{20\ 16\ 12\ 4^2\} + 35\{20\ 16\ 8^24\} \tag{4}$$

This looks encouraging BUT there exist counterexamples! One counter example is worth billions of examples!