Plethysm and Symplectic Models

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The use of the operation of plethysm in physics was recognised very early by Vladas Vanagas. This paper is dedicated to his memory.

Abstract

The implementation of symplectic models of nuclei and quantum dots requires the evaluation of plethysms of the fundamental irreducible representations of the non-compact group $Sp(2n,R)$. A method of making such evaluations is outlined and illustrated by a brief example.

1. Introduction

The non-compact symplectic group $Sp(2n,R)$ plays a key role in symplectic models of nuclei[1,2] and quantum dots[3−7]. An important problem in the implementation of the symplectic model is the resolution of symmetrised Kronecker powers[7−10] of $Sp(2n,R)$ into irreducible representations of $Sp(2n,R)$. Herein we outline a procedure for resolving such symmetrised powers which has recently been implemented in the computer package **SCHUR**[11].

2. The $Sp(2n,R) \to U(n)$ Branching Rule

The general problem of the $Sp(2n,R) \to U(n)$ reduction has been studied in some detailed[12,13] and has been implemented[6] in **SCHUR**. Under that restriction the two fundamental irreducible representations of $Sp(2n,R)$ decompose as

$$< \frac{1}{2}(0) > \to \varepsilon^{\frac{1}{2}}(\{0\} + \{2\} + \{4\} + \dots \tag{1}$$

$$< \frac{1}{2}(1) > \to \varepsilon^{\frac{1}{2}}(\{1\} + \{3\} + \{5\} + \dots \tag{2}$$

and in general

$$< \frac{k}{2}(\lambda) > \to \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda_s\}_N^k \cdot D_N\}_N \tag{3}$$

where $N = min(n,k)$, $\{\lambda_s\}^k$ is a *signed sequence*[12] of terms $\pm\{\rho\}$ such that $\pm[\rho]$ is equivalent to $[\lambda]$ under the modification rules[14−16] rules of the group $O(k)$, D_N is the infinite $S-$function series indexed by even partitions into not more than N parts. The first $\cdot$ indicates a product in $U(n)$ and the second $\cdot$ in $U(N)$ as implied by the final subscript N.

Equations (1) to (3) involve infinite series of irreducible representations of $U(N)$ and in any practical calculation must be truncated at some bound.

3. Evaluation of Plethysms for $Sp(2n,R)$

Central to our analysis is the fact that the labelling of the unitary positive discrete series of irreducible representations of $Sp(2n,R)$ is chosen so that under $Sp(2n,R) \to U(n)$ the lowest weight irreducible representation of $U(n)$ appearing in the reduction of $< \frac{k}{2}(\lambda) >$ is $\varepsilon^{\frac{k}{2}}\{\lambda\}$ with unit multiplicity. Noting the preceding we are led to the following algorithm for determining the $Sp(2n,R)$ content of an arbitrary $Sp(2n,R)$ plethysm $< \frac{k}{2}(\lambda) > \otimes\{\mu\}$ up to a chosen cutoff:-

Algorithm

1. *Evaluate the $U(n)$ plethysm $\{\lambda\} \otimes \{\mu\}$ up to terms of the maximum desired weight to produce a list (U) of $U(n)$ irreducible representations.*
2. *Let (S) be a null set of $Sp(2n,r)$ irreducible representations.*
3. *Select the set (W) of lowest weight $U(n)$ irreducible representations contained in the list U.*

4. *Associate each member $\{\rho\}$ of the set W with the $Sp(2n, R)$ irreducible representation $< \kappa(\rho) >$ where*
$$\kappa = \frac{k}{2} \times w_{\mu} \tag{4}$$
with w_{μ} being the weight of the partition (μ). These $Sp(2n, R)$ irreducible representations belong to a set $S1$. Put $S = S + S1$.
5. *Perform the reduction $Sp(2n, R) \to U(n)$ for the set $S1$ to produce a list $U1$ of $U(n)$ irreducible representations and put $U = U - U1$.*
6. *Repeat 3 to 5 until (U) becomes an empty set. Then the set (S) of $Sp(2n, R)$ irreducible representations is the desired result.*

4. An Example

In $n-$particle systems the $SU(2) \times Sp(6, R)$ plethysms of the type
$$\begin{aligned} &[(\{1\} < \frac{1}{2}; (0)) + (\{1\} < \frac{1}{2}; (1) >)] \otimes \{1^n\} \\ &= \sum_{r=0}^{n} [(\{1\} < \frac{1}{2}; (0)) \otimes \{1^{n-r}\} \cdot (\{1\} < \frac{1}{2}; (1) >)] \otimes \{1^r\} \end{aligned} \tag{5}$$
where $\{1\}$ is the basic representation of $SU(2)$, play an important role. To evaluate Eq.(5) we must be able to calculate plethysms of the type $< \frac{1}{2}; (0) > \otimes \{\lambda\}$ and $< \frac{1}{2}; (1) > \otimes \{\lambda\}$ In electron systems such as arise in three-dimensional quantum dots antisymmetrization restricts the $\{\lambda\}$ to partitions of the form $(2^s 1^r)$. For a six-particle system this would require the evaluation of plethysms involving partitions $(\lambda) = (0),\ (1),\ ...,\ (2^s 1^r)$ where $2s + r = 6$. By way of an example we give a tabulation of the $Sp(6, R)$ plethysms for $(\lambda = (2^3),\ (2^2 1^2),\ (21^4),\ (1^6)$ evaluated to weight 12 using the above algorithm.

$$\begin{aligned}< \tfrac{1}{2};(0) > \otimes\{2^3\} = {} & < 3;(12\ ) > + < 3;(11\ 1) > + 4 < 3;(10\ 2) > + < 3;(10\ 1^2) > \\ & + 3 < 3;(93) > + 4 < 3;(921) > + < 3;(91) > + 5 < 3;(84) > \\ & + 4 < 3;(831) > + 4 < 3;(82^2) > + < 3;(82) > + < 3;(81^2) > \\ & + < 3;(75) > + 4 < 3;(741) > + 2 < 3;(732) > + < 3;(73) > \\ & + 2 < 3;(721) > + 2 < 3;(6^2) > + 2 < 3;(651) > + 3 < 3;(642) > \\ & + < 3;(64) > + 2 < 3;(631) > + < 3;(62) > + < 3;(5^2) > \\ & + < 3;(541) > + < 3;(532) > + < 3;(521) > + < 3;(4^3) > \\ & + < 3;(4^2) > + < 3;(42^2) >\end{aligned}$$

$$\begin{aligned}< \tfrac{1}{2};(1) > \otimes\{2^3\} = {} & < 3;(10\ 2) > + 2 < 3;(10\ 1^2) > + 2 < 3;(93) > + 4 < 3;(921) > \\ & + < 3;(84) > + 4 < 3;(831) > + 2 < 3;(82^2) > + < 3;(82) > \\ & + 2 < 3;(75) > + 4 < 3;(741) > + 2 < 3;(732) > + 2 < 3;(721) > \\ & + 2 < 3;(651) > + < 3;(642) > + < 3;(64) > + < 3;(63^2) > \\ & + < 3;(631) > + 2 < 3;(62^2) > + 2 < 3;(5^2 2) > + < 3;(541) > \\ & + < 3;(521) > + < 3;(4^2 2) > + < 3;(2^3) >\end{aligned}$$

$$\begin{aligned}< \tfrac{1}{2};(0) > \otimes\{2^2 1^2\} = {} & 2 < 3;(11\ 1) > + 3 < 3;(10\ 2) > + 4 < 3;(10\ 1^2) > + 6 < 3;(93) > \\ & + 7 < 3;(921) > + < 3;(91) > + 4 < 3;(84) > + 9 < 3;(831) > \\ & + 2 < 3;(82^2) > + 2 < 3;(82) > + < 3;(81^2) > + 4 < 3;(75) > \\ & + 7 < 3;(741) > + 5 < 3;(732) > + 2 < 3;(73) > + 3 < 3;(721) > \\ & + 4 < 3;(651) > + 2 < 3;(642) > + 2 < 3;(64) > + 2 < 3;(63^2) > \\ & + 3 < 3;(631) > + 2 < 3;(62^2) > + < 3;(61^2) > + 2 < 3;(5^2 2) > \\ & + < 3;(543) > + 2 < 3;(541) > + < 3;(532) > + < 3;(53) > \\ & + < 3;(521) > + < 3;(4^2 2) > + < 3;(431) > + < 3;(3^2 2) >\end{aligned}$$

$$\begin{aligned}< \tfrac{1}{2};(1) > \otimes\{2^2 1^2\} = {} & 2 < 3;(10\ 2) > + 2 < 3;(10\ 1^2) > + 2 < 3;(93) > + 7 < 3;(921) > \\ & + 3 < 3;(84) > + 6 < 3;(831) > + 5 < 3;(82^2) > + < 3;(81^2) > \\ & + 2 < 3;(75) > + 7 < 3;(741) > + 3 < 3;(732) > + < 3;(73) > \\ & + 3 < 3;(721) > + < 3;(6^2) > + 4 < 3;(651) > + 5 < 3;(642) > \\ & + 2 < 3;(631) > + < 3;(62^2) > + < 3;(5^2) > + < 3;(543) > \\ & + 2 < 3;(541) > + < 3;(532) > + < 3;(521) > + < 3;(4^3) > \\ & + < 3;(42^2) >\end{aligned}$$

$$\begin{aligned}< \tfrac{1}{2};(0) > \otimes\{21^4\} = {} & < 3;(10\ 2) > + 2 < 3;(10\ 1^2) > + 2 < 3;(93) > + 3 < 3;(921) > \\ & + 2 < 3;(84) > + 5 < 3;(831) > + < 3;(82^2) > + < 3;(81^2) > \\ & + < 3;(75) > + 4 < 3;(741) > + 3 < 3;(732) > + < 3;(73) > \\ & + < 3;(721) > + < 3;(6^2) > + < 3;(651) > + 2 < 3;(642) > \\ & + < 3;(63^2) > + 2 < 3;(631) > + < 3;(543) > + < 3;(541) > \\ & + < 3;(532) > + < 3;(43^2) > + < 3;(431) >\end{aligned}$$

$$\begin{aligned}< \tfrac{1}{2};(1) > \otimes\{21^4\} = {} & < 3;(10\ 1^2) > + 3 < 3;(921) > + < 3;(84) > + 3 < 3;(831) > \\ & + < 3;(82^2) > + < 3;(75) > + 4 < 3;(741) > + 2 < 3;(732) > \\ & + < 3;(721) > + 2 < 3;(651) > + 2 < 3;(642) > + < 3;(631) > \\ & + < 3;(62^2) > + < 3;(543) > + < 3;(541) > + < 3;(4^2 2) >\end{aligned}$$

$$\begin{aligned}< \tfrac{1}{2};(0) > \otimes\{1^6\} = {} & < 3;(831) > + < 3;(741) > + < 3;(732) > + < 3;(63^2) > \\ & + < 3;(4^2 2) >\end{aligned}$$

$$< \tfrac{1}{2};(1) > \otimes\{1^6\} = < 3;(82^2) > + < 3;(741) > + < 3;(642) > + < 3;(4^3) >$$

5. Concluding Remark

The above algorithm gives a systematic way of evaluating the type of plethysms required in implementations of symplectic models. As with all types of plethysms the problem is combinatorially explosive as the number of particles increases.

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