Notes on $PSL(2,11) \supset \mathcal{I}$

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4 July 2003

Abstract. These notes are compiled as my attempt to understand some of the properties of the embedding of the icosahedral group, $\mathcal{I}$, in the finite group $PSL(2,11)$.

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1. Introduction

These notes are compiled as my attempt to understand some of the properties of the embedding of the icosahedral group in the finite group $PSL(2,11)$

1.1. Characters of $PSL(2,11)$

Table 1 The Characters of$PSL(2,11)$.

	(1^{11})	$55(2^4 1^3)$	$110(3^3 1^2)$	$132(5^2 1)_+$	$132(5^2 1)_-$	$110(632)$	$60(11\ )_+$	$60(11\ )_-$
χ_1	1	1	1	1	1	1	1	1
χ_2	5	1	-1	0	0	1	β	β^*
χ_3	5	1	-1	0	0	1	β^*	β
χ_4	10	-2	1	0	0	1	-1	-1
χ_5	10	2	1	0	0	-1	-1	-1
χ_6	11	-1	-1	1	1	-1	0	0
χ_7	12	0	0	α	α'	0	1	1
χ_8	12	0	0	α'	α	0	1	1

$$\alpha = \tfrac{1}{2}(-1+\sqrt{5}),\ \alpha' = -\tfrac{1}{2}(1+\sqrt{5}),\ \beta = \tfrac{1}{2}(-1+i\sqrt{11})$$

NB. We have used the partitions appropriate to the classes of the symmetric group S_{11}. The group $PSL(2,11)$ is a subgroup of S_{11}. Note that the conjugacy classes all derive from *even* permutations which are appropriate to the group A_{11}.

1.2. Characters of the Icosahedral Group

Chemists commonly present the character table as below

Table 2 Characters of the ordinary irreducible representations of I.

	1	$20C_3$	$15C_2$	$12C_5$	$12C_5^2$
A	1	1	1	1	1
T_1	3	0	-1	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
T_2	3	0	-1	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$
U	4	1	0	-1	-1
V	5	-1	1	0	0

while in terms of the isomorphic alternating group A_5 they can be written as

Table 3 Characters of the ordinary irreducible representations of A_5.

	(1^5)	$20(31^2)$	$15(2^2 1)$	$12(5)_+$	$12(5)_-$
$[5]$	1	1	1	1	1
$[31^2]_+$	3	0	-1	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
$[31^2]_-$	3	0	-1	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$
$[41]$	4	1	0	-1	-1
$[32]$	5	-1	1	0	0

NB. We designate irreducible representations of A_n by partitions of n enclosed in square brackets, $[,]$, and distinguish splitting irreducible representations by attaching a $\pm$ as a subscript.

1.3. The $PSL(2,11) \to \mathcal{I}$ Branching Rules

These follow from the character tables of the respective groups to give
Table 4 The $PSL(2,11) \to \mathcal{I}$ Branching Rules.

$$\begin{array}{lcl}
\chi_1 & \to & A \\
\chi_2 & \to & V \\
\chi_3 & \to & V \\
\chi_4 & \to & T_1 + T_2 + U \\
\chi_5 & \to & A + U + V \\
\chi_6 & \to & T_1 + T_2 + V \\
\chi_7 & \to & T_1 + U + V \\
\chi_8 & \to & T_2 + U + V
\end{array}$$

1.4. Kronecker Products for Ordinary Representations of I

Table 5 Kronecker products for the ordinary irreducible representations of I.

$$\begin{array}{c} \\ A \\ T_1 \\ T_2 \\ U \\ V \end{array}
\begin{array}{c}
\begin{array}{ccccc} A & T_1 & T_2 & U & V \end{array} \\
\left(\begin{array}{ccccc}
A & T_1 & T_2 & U & V \\
T_1 & \{A+V\}+[T_1] & U+V & T_2+U+V & T_1+T_2+U+V \\
T_2 & U+V & \{A+V\}+[T_2] & T_1+U+V & T_1+T_2+U+V \\
U & T_2+U+V & T_1+U+V & \{A+U+V\}+[T_1+T_2] & T_1+T_2+U+2V \\
V & T_1+T_2+U+V & T_1+T_2+U+V & T_1+T_2+U+2V & \{A+U+2V\}+[T_1+T_2+U]
\end{array}\right)
\end{array}$$

1.5. Kronecker Squares for the representations of $PSL(2,11)$

Below we give the resolution of the Kronecker squares of the irreducible representations of $PSL(2,11)$ into their symmetric and antisymmetric terms.
Table 6 Resolution of the Kronecker squares of the irreducible representations of $PSL(2,11)$.

χ_i	$\chi_i \otimes \{2\}$	$\chi_i \otimes \{1^2\}$
χ_1	χ_1	$-$
χ_2	$\chi_3 + \chi_5$	χ_4
χ_3	$\chi_2 + \chi_5$	χ_4
χ_4	$\chi_1 + \chi_2 + \chi_3 + 2\chi_5 + \chi_7 + \chi_8$	$\chi_4 + \chi_6 + \chi_7 + \chi_8$
χ_5	$\chi_1 + \chi_2 + \chi_3 + 2\chi_5 + \chi_7 + \chi_8$	$\chi_4 + \chi_6 + \chi_7 + \chi_8$
χ_6	$\chi_1 + \chi_2 + \chi_3 + 2\chi_5 + \chi_6 + \chi_7 + \chi_8$	$2\chi_4 + \chi_6 + \chi_7 + \chi_8$
χ_7	$\chi_1 + \chi_2 + \chi_3 + 2\chi_5 + \chi_6 + \chi_7 + 2\chi_8$	$2\chi_4 + 2\chi_6 + \chi_7 + \chi_8$
χ_8	$\chi_1 + \chi_2 + \chi_3 + 2\chi_5 + \chi_6 + 2\chi_7 + \chi_8$	$2\chi_4 + 2\chi_6 + \chi_7 + \chi_8$

Table 6a Resolution of non-trivial Kronecker products of irreducible representations of $PSL(2,11)$.

$$
\begin{aligned}
\chi_2 \times \chi_3 &= \chi_1 + \chi_7 + \chi_8 \\
\chi_2 \times \chi_4 &= \chi_3 + \chi_4 + \chi_6 + \chi_7 + \chi_8 \\
\chi_2 \times \chi_5 &= \chi_3 + \chi_5 + \chi_6 + \chi_7 + \chi_8 \\
\chi_2 \times \chi_6 &= \chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8 \\
\chi_2 \times \chi_7 &= \chi_2 + \chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8 \\
\chi_2 \times \chi_8 &= \chi_2 + \chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8 \\
\chi_3 \times \chi_4 &= \chi_2 + \chi_4 + \chi_6 + \chi_7 + \chi_8 \\
\chi_3 \times \chi_5 &= \chi_2 + \chi_5 + \chi_6 + \chi_7 + \chi_8 \\
\chi_3 \times \chi_6 &= \chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8 \\
\chi_3 \times \chi_7 &= \chi_3 + \chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8 \\
\chi_3 \times \chi_8 &= \chi_3 + \chi_4 + \chi_5 + \chi_6 + \chi_7 + \chi_8 \\
\chi_4 \times \chi_5 &= 2\chi_4 + \chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8 \\
\chi_4 \times \chi_6 &= \chi_2 + \chi_3 + \chi_4 + 2\chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8 \\
\chi_4 \times \chi_7 &= \chi_2 + \chi_3 + 2\chi_4 + 2\chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8 \\
\chi_4 \times \chi_8 &= \chi_2 + \chi_3 + 2\chi_4 + 2\chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8 \\
\chi_5 \times \chi_6 &= \chi_2 + \chi_3 + 2\chi_4 + \chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8 \\
\chi_5 \times \chi_7 &= \chi_2 + \chi_3 + 2\chi_4 + 2\chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8 \\
\chi_5 \times \chi_8 &= \chi_2 + \chi_3 + 2\chi_4 + 2\chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8 \\
\chi_6 \times \chi_7 &= \chi_2 + \chi_3 + 2\chi_4 + 2\chi_5 + 2\chi_6 + 3\chi_7 + 2\chi_8 \\
\chi_6 \times \chi_8 &= \chi_2 + \chi_3 + 2\chi_4 + 2\chi_5 + 2\chi_6 + 2\chi_7 + 3\chi_8 \\
\chi_7 \times \chi_8 &= \chi_2 + \chi_3 + 2\chi_4 + 2\chi_5 + 2\chi_6 + 3\chi_7 + 3\chi_8
\end{aligned}
$$

Inspection of the above results allows us to conclude that the irreps χ_2, χ_3 form a complex pair while the remaining irreducible representations are all real and orthogonal.

1.6. Embedding $PSL(2,11)$ in S_{11}

The irreducible representation $\{10\ 1\}$ of the symmetric group S_{11} is of dimension 10 which is real and orthogonal, though NOT unimodular. We note that the symmetrized square of this irreducible representation may be resolved as

$$\{10\ 1\} \otimes \{2\} = \{11\} + \{10\ 1\} + \{92\} \tag{1}$$

$$\{10\ 1\} \otimes \{1^2\} = \{91^2\} \tag{2}$$

Three possible embeddings in $\{10\ 1\}$ might be considered

$$\{10\ 1\} \rightarrow \chi_2 + \chi_3 \tag{3a}$$

$$\{10\ 1\} \rightarrow \chi_4 \tag{3b}$$

$$\{10\ 1\} \rightarrow \chi_5 \tag{3c}$$

The embedding (3b) is inconsistent with (1) since the symmetric square of χ_4 does not contain itself. In the case of (3a) we are immediately led to the decompositions

$$\{11\} \to \chi_1 \tag{4a}$$
$$\{10\,1\} \to \chi_2 + \chi_3 \tag{4b}$$
$$\{92\} \to 2\chi_5 + \chi_7 + \chi_8 \tag{4c}$$
$$\{91^2\} \to \chi_1 + 2\chi_4 + \chi_7 + \chi_8 \tag{4d}$$

In the case of (3c) we are led to the decompositions

$$\{11\} \to \chi_1 \tag{5a}$$
$$\{10\,1\} \to \chi_5 \tag{5b}$$
$$\{92\} \to \chi_2 + \chi_3 + \chi_5 + \chi_7 + \chi_8 \tag{5c}$$
$$\{91^2\} \to \chi_4 + \chi_6 + \chi_7 + \chi_8 \tag{5d}$$

These branching rules can be readily extended by using the S_{11} characters evaluated over the classes of $PSL(2,11)$.

1.7. $S_{11} \to PSL(2,11)$ Branching Rules

Below we give the relevant $S_{11} \to PSL(2,11)$ decompositions for the embedding defined by (3c). Since under $S_{11} \to PSL(2,11)$ the decomposition of the irreducible representations labelled by conjugate partitions are the same we give only the decomposition in the case of only one partition of each conjugate pair. Note that in S_{11} the two irreducible representations $\{61^5\}$ and $\{43^21\}$ are self-conjugate. Recall that under $S_n \to A_n$ the character associated with a self-conjugate irreducible representation of S_n is the sum of two simple characters of A_n. These characters of A_n take exactly half the values of the characteristics of S_n save for the splitting classes of A_n. In the case of A_{11} the splitting classes are $(11\ )_\pm$ and $(731)_\pm$.

Table 7 Branching rules for $S_{11} \to PSL(2,11)$.

$Dim(\lambda)$	$\{\lambda\}$	Decomposition
1	$\{11\}$	χ_1
10	$\{10\ 1\}$	χ_5
44	$\{92\}$	$\chi_2+\chi_3+\chi_5+\chi_7+\chi_8$
45	$\{91^2\}$	$\chi_4+\chi_6+\chi_7+\chi_8$
110	$\{83\}$	$\chi_1+\chi_2+\chi_3+\chi_4+3\chi_5+\chi_6+2\chi_7+2\chi_8$
231	$\{821\}$	$2\chi_2+2\chi_3+3\chi_4+3\chi_5+5\chi_6+4\chi_7+4\chi_8$
120	$\{81^3\}$	$4\chi_4+\chi_5+2\chi_6+2\chi_7+2\chi_8$
165	$\{74\}$	$\chi_1+\chi_2+\chi_3+2\chi_4+4\chi_5+2\chi_6+3\chi_7+3\chi_8$
550	$\{731\}$	$\chi_1+4\chi_2+4\chi_3+9\chi_4+8\chi_5+9\chi_6+10\chi_7+10\chi_8$
385	$\{72^2\}$	$\chi_1+4\chi_2+4\chi_3+4\chi_4+7\chi_5+6\chi_6+7\chi_7+7\chi_8$
594	$\{721^2\}$	$4\chi_2+4\chi_3+10\chi_4+8\chi_5+10\chi_6+11\chi_7+11\chi_8$
210	$\{71^4\}$	$\chi_1+\chi_2+\chi_3+3\chi_4+4\chi_5+3\chi_6+4\chi_7+4\chi_8$
132	$\{65\}$	$\chi_1+2\chi_2+2\chi_3+\chi_4+2\chi_5+3\chi_6+2\chi_7+2\chi_8$
693	$\{641\}$	$5\chi_2+5\chi_3+11\chi_4+10\chi_5+11\chi_6+13\chi_7+13\chi_8$
990	$\{632\}$	$2\chi_1+8\chi_2+8\chi_3+14\chi_4+16\chi_5+16\chi_6+18\chi_7+18\chi_8$
1232	$\{631^2\}$	$\chi_1+8\chi_2+8\chi_3+21\chi_4+16\chi_5+23\chi_6+22\chi_7+22\chi_8$
1100	$\{62^21\}$	$3\chi_1+9\chi_2+9\chi_3+15\chi_4+19\chi_5+17\chi_6+20\chi_7+20\chi_8$
924	$\{621^3\}$	$\chi_1+8\chi_2+8\chi_3+13\chi_4+14\chi_5+15\chi_6+17\chi_7+17\chi_8$
252	$\{61^5\}$	$2\chi_1+3\chi_2+3\chi_3+2\chi_4+6\chi_5+4\chi_6+4\chi_7+4\chi_8$
330	$\{5^21\}$	$3\chi_2+3\chi_3+4\chi_4+5\chi_5+6\chi_6+6\chi_7+6\chi_8$
990	$\{542\}$	$2\chi_1+8\chi_2+8\chi_3+14\chi_4+16\chi_5+16\chi_6+18\chi_7+18\chi_8$
1155	$\{541^2\}$	$2\chi_1+8\chi_2+8\chi_3+19\chi_4+17\chi_5+19\chi_6+21\chi_7+21\chi_8$
660	$\{53^2\}$	$\chi_1+4\chi_2+4\chi_3+11\chi_4+10\chi_5++11\chi_6+12\chi_7+12\chi_8$
2310	$\{5321\}$	$3\chi_1+18\chi_2+18\chi_3+35\chi_4+34\chi_5+39\chi_6+42\chi_7+42\chi_8$
1540	$\{531^3\}$	$2\chi_1+11\chi_2+11\chi_3+24\chi_4+23\chi_5+26\chi_6+28\chi_7+28\chi_8$
825	$\{52^3\}$	$3\chi_1+6\chi_2+6\chi_3+12\chi_4+15\chi_5+12\chi_6+15\chi_7+15\chi_8$
462	$\{4^23\}$	$2\chi_1+3\chi_2+3\chi_3+8\chi_4+7\chi_5+8\chi_6+8\chi_7+8\chi_8$
1320	$\{4^221\}$	$2\chi_1+11\chi_2+11\chi_3+18\chi_4+21\chi_5+22\chi_6+22\chi_7+22\chi_8$
1188	$\{43^21\}$	$8\chi_2+8\chi_3+20\chi_4+16\chi_5+20\chi_6+22\chi_7+22\chi_8$

Under $S_{11} \to A_{11}$ the above irreducible representations are irreducible apart from the two irreducible representations $\{61^5\}$ and $\{43^21\}$ which split as

$$\{61^5\} \to [61^5]_+ + [61^5]_-$$

$$\{43^21\} \to [43^21]_+ + [43^21]_-$$

Under $A_{11} \to PSL(2,11)$ the branching rules are the same as in the above Table apart from the splitting cases where

$$[61^5]_+ \to \chi_1 + \chi_2 + 2\chi_3 + \chi_4 + 3\chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8$$
$$[61^5]_- \to \chi_1 + 2\chi_2 + \chi_3 + \chi_4 + 3\chi_5 + 2\chi_6 + 2\chi_7 + 2\chi_8$$

$$[43^21]_+ \to 4\chi_2 + 4\chi_3 + 10\chi_4 + 8\chi_5 + 10\chi_6 + 11\chi_7 + 11\chi_8$$
$$[43^21]_- \to 4\chi_2 + 4\chi_3 + 10\chi_4 + 8\chi_5 + 10\chi_6 + 11\chi_7 + 11\chi_8$$

1.8. Embeddings of $PSL(2,11)$ in the Orthogonal groups O_n and SO_n

The group S_{n+1} may be embedded in the orthogonal group O_n. In that case the $n-$dimensional irreducible representation $\{n,1\}$ is embedded in the vector irreducible representation [1] of the full orthogonal group, $O(n)$. The irreducible representation $\{n,1\}$ is orthogonal but not unimodular and hence S_{n+1} cannot be embedded in the special orthogonal group SO_n. For $n \geq 5$ all the irreducible representations of the alternating group are both orthogonal and unimodular and as such can be embedded in an appropriate SO_n. For example, in the case of the icosahedral group $\mathcal{I} \sim A_5$ we may embed the irreducible representation U in the vector irreducible representation [10] of SO_4 or the irreducible representation V in the vector irreducible representation of SO_5 etc. For the group $PSL(2,11)$ the irreducible representations $\chi_2 + \chi_3$ and χ_5 are orthogonal and unimodular and hence may be embedded in SO_{10}. However, it is sufficient to note that $SO_{10} \supset A_{11}$ and one has the typical $SO_{10} \to A_{11}$ decompositions

Table 8 Some $SO_{10} \to A_{11}$ branching rules

$Dim(\lambda)$	$[\lambda]$	Decomposition
1	[0]	[11]
10	[1]	[10 1]
54	[2]	[10 1] + [92]
45	$[1^2]$	$[91^2]$
210	[3]	[11] + [10 1] + [92] + $[91^2]$
320	[21]	[92] + $[91^2]$ + [821]
120	$[1^3]$	$[81^3]$
660	[4]	[11] + 2[10 1] + 2[92] + $[91^2]$ + [83] + [821] + [74]
1386	[31]	[10 1] + [92] + 2$[91^2]$ + [83] + 2[821] + $[81^3]$ + [731]
770	$[2^2]$	[92] + [83] + [821] + $[72^2]$
945	$[21^2]$	[821] + $[81^3]$ + $[721^2]$
210	$[1^4]$	$[71^4]$

We note that in a similar fashion we have $SO_4 \supset A_5$ and $SO_7 \supset A_8$. In the latter case while $A_8 \supset L_{168}$ one does not have $G_2 \supset A_8$. In the former case it may be naturally seen from the fact that the pentahedral group P is a subgroup of $SO(4)$ as noted elsewhere[4,5].

1.9. Spin irreducible representations of A_n

The alternating groups A_n possess *spin* or *projective* irreducible representations which may be labelled by partitions of n into k distinct parts[1−3] with the understanding that irreducible representations with $n-k$ *even* constitute a pair of conjugate irreducible representations. Noting that $SO_{n-1} \supset A_n$ we may equivalently label the projective irreducible representations of A_n as $[\Delta;\lambda]$ where λ is a partition into p distinct parts with the weight, ω_λ, of λ is such that $n-\omega_\lambda > \lambda_1$ and $\lambda_1 < [\frac{n}{2}]$. In that case if the partition (λ) is such that $n-p$ is *odd* then the irreducible representation forms a conjugate pair and the two irreducible representations are distinguished by attaching a $\pm$ as a subscript to $[\Delta;\lambda]$. Thus for A_5 we have the projective irreducible representations and their dimensions as given below

Table 9 Dimensions of the projective irreducible representations of A_5.

Dim$[\Delta;\lambda]$	*Irreducible representation*
2	$[\Delta;0]_\pm$
6	$[\Delta;1]$
4	$[\Delta;2]$

while for A_8 we have

Table 10 Dimensions of the projective irreducible representations of A_8.

Dim$[\Delta;\lambda]$	*Irreducible representation*
8	$[\Delta;0]$
24	$[\Delta;1]_\pm$
56	$[\Delta;2]_\pm$
64	$[\Delta;21]$
56	$[\Delta;3]_\pm$
48	$[\Delta;31]$

and finally for A_{11} we have

Table 11 Dimensions of the projective irreducible representations of A_{11}.

Dim$[\Delta;\lambda]$	*Irreducible representation*
16	$[\Delta;0]_\pm$
144	$[\Delta;1]$
560	$[\Delta;2]$
616	$[\Delta;21]_\pm$
1200	$[\Delta;3]$
1584	$[\Delta;31]_\pm$
1232	$[\Delta;32]_\pm$
528	$[\Delta;321]$
1440	$[\Delta;4]$
1584	$[\Delta;41]_\pm$
880	$[\Delta;42]_\pm$
672	$[\Delta;5]$

1.10. $PSL(2,11)$ as a subgroup of SU_5

The group $PSL(2,11)$ is a natural subgroup of the special unitary group SU_5 with the vector irreducible representation of SU_5, $\{1\}$, branching to the complex χ_2 irreducible representation of $PSL(2,11)$. The $SU_5 \to PSL(2,11)$ branching rules may be readily evaluated and a typical table is given below.

Table 12 The $SU_5 \to PSL(2,11)$ branching rules.

$Dim(\lambda)$	SU_5	$PSL(2,11)$
1	$\{0\}$	χ_1
5	$\{1\}$	χ_2
15	$\{2\}$	$\chi_3+\chi_5$
10	$\{1^2\}$	χ_4
35	$\{3\}$	$\chi_1+\chi_5+\chi_7+\chi_8$
40	$\{21\}$	$\chi_3+\chi_6+\chi_7+\chi_8$
10	$\{1^3\}$	χ_4
70	$\{4\}$	$2\chi_2+\chi_3+2\chi_5+\chi_6+\chi_7+\chi_8$
105	$\{31\}$	$\chi_2+2\chi_4+\chi_5+2\chi_6+2\chi_7+2\chi_8$
50	$\{2^2\}$	$\chi_1+\chi_2+2\chi_5+\chi_7+\chi_8$
45	$\{21^2\}$	$\chi_4+\chi_6+\chi_7+\chi_8$
5	$\{1^4\}$	χ_3
126	$\{5\}$	$\chi_1+\chi_2+2\chi_3+\chi_4+3\chi_5+2\chi_6+2\chi_7+2\chi_8$
224	$\{41\}$	$\chi_2+2\chi_3+4\chi_4+4\chi_5+3\chi_6+4\chi_7+4\chi_8$
175	$\{32\}$	$\chi_2+2\chi_3+2\chi_4+3\chi_5+3\chi_6+3\chi_7+3\chi_8$
126	$\{31^2\}$	$\chi_2+3\chi_4+\chi_5+3\chi_6+2\chi_7+2\chi_8$
75	$\{2^21\}$	$\chi_2+\chi_3+\chi_4+2\chi_5+\chi_6+\chi_7+\chi_8$
24	$\{21^3\}$	$\chi_7+\chi_8$
1	$\{1^5\}$	χ_1
210	$\{6\}$	$2\chi_1+2\chi_2+2\chi_3+\chi_4+6\chi_5+2\chi_6+4\chi_7+4\chi_8$
420	$\{51\}$	$3\chi_2+3\chi_3+7\chi_4+4\chi_5+8\chi_6+8\chi_7+8\chi_8$
420	$\{42\}$	$2\chi_1+4\chi_2+4\chi_3+5\chi_4+7\chi_5+6\chi_6+8\chi_7+8\chi_8$
280	$\{41^2\}$	$\chi_2+2\chi_3+5\chi_4+4\chi_5+5\chi_6+5\chi_7+5\chi_8$
175	$\{3^2\}$	$6\chi_4+\chi_5+3\chi_6+3\chi_7+3\chi_8$
280	$\{321\}$	$2\chi_2+3\chi_3+3\chi_4+5\chi_5+5\chi_6+5\chi_7+5\chi_8$
70	$\{31^3\}$	$\chi_2+2\chi_4+\chi_5+\chi_6+\chi_7+\chi_8$
50	$\{2^3\}$	$\chi_1+\chi_3+2\chi_5+\chi_7+\chi_8$
45	$\{2^21^2\}$	$\chi_4+\chi_6+\chi_7+\chi_8$
5	$\{21^4\}$	χ_2

Table 12 The $SU_5 \to PSL(2,11)$ branching rules (continued).

$Dim(\lambda)$	SU_5	$PSL(2,11)$
330	$\{7\}$	$\chi_1+4\chi_2+4\chi_3+2\chi_4+7\chi_5+5\chi_6+6\chi_7+6\chi_8$
720	$\{61\}$	$\chi_1+6\chi_2+5\chi_3+11\chi_4+11\chi_5+12\chi_6+13\chi_7+13\chi_8$
840	$\{52\}$	$2\chi_1+7\chi_2+6\chi_3+12\chi_4+15\chi_5+13\chi_6+15\chi_7+15\chi_8$
540	$\{51^2\}$	$3\chi_2+3\chi_3+10\chi_4+6\chi_5+10\chi_6+10\chi_7+10\chi_8$
560	$\{43\}$	$5\chi_2+5\chi_3+9\chi_4+7\chi_5+10\chi_6+10\chi_7+10\chi_8$
700	$\{421\}$	$2\chi_1+6\chi_2+5\chi_3+10\chi_4+11\chi_5+11\chi_6+13\chi_7+13\chi_8$
160	$\{41^3\}$	$\chi_2+2\chi_3+2\chi_4+2\chi_5+3\chi_6+3\chi_7+3\chi_8$
315	$\{3^21\}$	$\chi_2+2\chi_3+6\chi_4+3\chi_5+6\chi_6+6\chi_7+6\chi_8$
210	$\{32^2\}$	$\chi_1+2\chi_2+2\chi_3+2\chi_4+4\chi_5+3\chi_6+4\chi_7+4\chi_8$
175	$\{321^2\}$	$\chi_2+\chi_3+3\chi_4+3\chi_5+3\chi_6+3\chi_7+3\chi_8$
15	$\{31^4\}$	$\chi_3+\chi_5$
40	$\{2^31\}$	$\chi_2+\chi_6+\chi_7+\chi_8$
10	$\{2^21^3\}$	χ_4

1.11. $SO(10) \to PSL(2,11)$ *Decompositions*

Table 13 Some $SO(10) \to PSL(2,11)$ Decompositions

$Dim[\lambda]$	$SO(10)$	$PSL(2,11)$
1	$[0]$	χ_1
10	$[1]$	$\chi_2+\chi_3$
45	$[1^2]$	$\chi_1+2\chi_4+\chi_7+\chi_8$
54	$[2]$	$\chi_2+\chi_3+2\chi_5+\chi_7+\chi_8$
210	$[3]$	$\chi_1+\chi_2+\chi_3+4\chi_4+4\chi_5+2\chi_6+4\chi_7+4\chi_8$
320	$[21]$	$3\chi_2+3\chi_3+6\chi_4+2\chi_5+6\chi_6+6\chi_7+6\chi_8$
120	$[1^3]$	$\chi_2+\chi_3+4\chi_4+2\chi_6+2\chi_7+2\chi_8$
660	$[4]$	$\chi_1+8\chi_2+8\chi_3+5\chi_4+12\chi_5+11\chi_6+12\chi_7+12\chi_8$
1386	$[31]$	$\chi_1+10\chi_2+10\chi_3+21\chi_4+20\chi_5+25\chi_6+25\chi_7+25\chi_8$
770	$[2^2]$	$4\chi_1+6\chi_2+6\chi_3+10\chi_4+16\chi_5+10\chi_6+14\chi_7+14\chi_8$
945	$[21^2]$	$6\chi_2+6\chi_3+17\chi_4+12\chi_5+17\chi_6+17\chi_7+17\chi_8$
210	$[1^4]$	$\chi_1+3\chi_2+3\chi_4+2\chi_5+3\chi_6+4\chi_7+3\chi_8$

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