Recent developments concerning non-compact groups and their properties

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And yet the mystery of mysteries is to view machines making machines; a spectacle that fills the mind with curious, and even awful, speculation.
— Benjamin Disraeli: *Coningsby* (1844)

ABSTRACT

The non-compact group $Sp(2n,\Re)$ plays an important role in symplectic many-body problems in physics such as arise in harmonic oscillator based models of nuclei and quantum dots while the non-compact group $U(p,q)$ is relevant to Coulomb type problems. Few applications are known in the case of the non-compact group $SO^*(2n)$. The non-trivial unitary irreducible representations of these groups are all of infinite dimension. We summarise recent work on the tensor products, plethysms and branching rules associated with these groups and relationships between these non-compact groups.

1. Introduction

The study of non-compact groups and their relationship to physics has a long history. In Buchheim's papers[1-4] of the 1880's on the theory of screws and wrenches, and in Study's work in 1903 on the electric dynamo[5] we recognise some of the properties of the Lorentz group while in Cunningham's[6] analysis of the symmetry properties of the source free Maxwell equations and in Bateman's[7] studies of optics we see the entrance of the conformal groups. The physicists interest in Lie groups largely springs from the early work of Weyl[8] and van der Waerden[9] and Yamanouchi[10-12]. The initial interest was in the compact Lie groups especially in the pioneering work of Giulio Racah[13,14] which was rapidly taken up by the nuclear physicists[15,16] and only much later by the atomic spectroscopists[17].

Physicist's interest in non-compact groups largely arose from the seminal papers of Wigner[18] and Bargmann[19] on the Lorentz group and Pauli's review[20]. Pauli[21] had early noticed the role of the compact Lie group $SO(4)$ as the degeneracy group of the H-atom. The existence of $SU(3)$ as the degeneracy group of the isotropic three-dimensional harmonic oscillator was used by Elliott[22] in his $SU(3)$ nuclei model. These Lie groups, or more correctly, Lie algebras, allowed one to ladder between the states associated with a given degenerate level but not between states belonging to different degenerate levels. This deficiency is overcome when one enlarges the Lie description of the system to include operators that ladder between different sets of degenerate states. In the case of the H-atom the *dynamical group* is the non-compact group $SO(4,2) \sim SU(2,2)$ group[23] and in the case of the isotropic harmonic oscillator the non-compact symplectic $Sp(6,\Re)$ group[24].

Here I propose to consider three classes of non-compact groups - $Sp(2n,\Re)$, $U(p,q)$ and $SO^*(2n)$. The first two groups are relevant to many-particle harmonic oscillator and Coulomb problems respectively. The group $SO^*(2n)$ leaves invariant the form

$$-z_1 z_{n+1}^* + z_{n+1} z_1^* - \cdots - z_n z_{2n}^* + z_{2n} z_n^*$$

The application of $SO^*(2n)$ to physical problems appears to be obscure but as we shall see it is intimately related to the other two classes of groups all of which have an appropriate metaplectic group $Mp(N)$ as their covering group.

My task is made easier by following upon Prof. R. C. King's presentation where much of the basic theory was outlined and will not be repeated here.

The sagacious reader who is capable of reading between these lines what does not stand written in them, but is nevertheless implied, will be able to form some conception

— Goethe

2. Discrete harmonic series representations

The non-compact semisimple Lie groups are characterized by finite non-unitary and infinite dimensional unitary irreducible representations. Of the latter, we shall restrict ourselves to the so-called discrete harmonic series of irreducible representations where the weights are bounded from below but not from above. The groups $Sp(2n,\Re)$ and $O(k)$ form a *dual pair* with respect to the metaplectic group $Mp(2nk)$ such that the basic irreducible representation $\tilde{\Delta}$ under $Sp(2nk,\Re)\to Sp(2n,\Re)\times O(k)$ branches as[25-27]

$$\tilde{\Delta}\to\sum_{\lambda}\langle\tfrac{1}{2}k(\lambda)\rangle\times[\lambda] \tag{2.1}$$

where the summation is over all λ such that

$$\lambda_1'+\lambda_2'\leq k \quad\text{and}\quad \lambda_1'\leq n$$

Likewise, for the dual pair $SO^*(2n),\ Sp(2k)$ we have[28]

$$\tilde{\Delta}\to\sum_{\lambda}[k(\lambda)]\times\langle\lambda\rangle \tag{2.2}$$

where the summation is over all λ such that

$$\lambda_1'\leq\min(n,k)$$

Finally, we have the dual pair $U(p,q),\ U(k)$ with[26,29]

$$\tilde{\Delta}\to\sum_{\nu,\mu}\{k(\bar{\nu};\mu)\}\times\{\bar{\nu};\mu\} \tag{2.3}$$

where the summation is carried out over all pairs of partitions (μ) and (ν) for which the conjugate partitions (ν') and (μ') satisfy the constraints

$$\mu_1'+\nu_1'\leq k,\quad \mu_1'\leq p\quad\text{and}\quad \nu_1'\leq q$$

N.B. The reduction in all three cases is multiplicity free.

Under $Mp(2n)\to Sp(2n,\Re)$ we have[25,27]

$$\tilde{\Delta}\to\langle\tfrac{1}{2}(0)\rangle+\langle\tfrac{1}{2}(1)\rangle \tag{2.4}$$

with $\langle\frac{1}{2}(0)\rangle$ and $\langle\frac{1}{2}(1)\rangle$ being termed the fundamental irreducible representations of $Sp(2n,\Re)$.

Under $Mp(2n)\to SO^*(2n)$ we have[28]

$$\tilde{\Delta}\to\sum_{m=0}^{\infty}[1(m)] \tag{2.5}$$

It is convenient to designate the infinite set of fundamental irreducible representations of $SO^*(2n)$ as

$$H = \sum_{m=0}^{\infty} H_m = \sum_{m=0}^{\infty} [1(m)] \tag{2.6}$$

and write $H = H_+ + H_-$ with

$$H_+ = \sum_{k=0}^{\infty} [1(2k)] \quad \text{and} \quad H_- = \sum_{k=0}^{\infty} [1(2k+1)]. \tag{2.7}$$

Finally, under $Mp(2p+2q) \to U(p,q)$ we have[25,30]

$$\tilde{\Delta} \to H = H_0 + \sum_{m=1}^{\infty} (H_m + H_{-m}) \tag{2.8}$$

where

$$\begin{aligned} H_0 \ &= \{1(\bar{0};0)\} && (2.9a) \\ H_m \ &= \{1(\bar{0};m)\} \quad m = 1,2,\ldots && (2.9b) \\ H_{-m} &= \{1(\bar{m};0)\} \quad m = 1,2,\ldots && (2.9c) \end{aligned}$$

3. What do we need to know?

Among the requirements that we need in order to make practical calculations we may list:

1. Group-subgroup decompositions.
2. Tensor product decompositions.
3. Resolution of symmetrised tensor products (plethysms).
4. Relationship between the various group-subgroup chains.
5. Resolution of tensor products of finite non-unitary with infinite discrete unitary representations.
6. Complete classification of all representations.
7. Calculation of matrix elements.

While there has been considerable progress in recent years on all the above topics I shall only touch on some of them. Topic 5 is the subject of Toumazet's contribution to this volume.

4. Branching rules for non-compact Lie groups

Decompositions to maximal compact subgroups involve an infinite set of irreducible representations of the subgroup. Thus under $Sp(2n,\Re) \to U(n)$ we have[25-27]

$$\langle \tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda_s\}_N^{[k]} \cdot D_N\}_N \tag{4.1}$$

where $N = \min(n,k)$.

Likewise, under $So^*(2n) \to U(n)$ we have[28]

$$[k(\lambda)] \to \varepsilon^{k} \cdot \{\{\lambda_s\}_N^{\langle 2k\rangle} \cdot B_N\}_N \tag{4.2}$$

with $N = \min(n,2k)$.

Similar results can be obtained for $U(p,q) \to U(q) \times U(p)$[26,31].

In each case the results involve infinite series of special Schur functions (S-functions), certain signed sequences of S-functions[25] whose products are evaluated using the Littlewood-Richardson rule. Explicit calculations are usually quite tedious and best left to computer implementations[31].

5. Symmetrised tensor products

Methods of computing tensor products of discrete series harmonic irreducible representations are now well established[25-30] and capable of computation up to a user defined cutoff[31]. While such tensor products can be useful in practical applications of greater interest is the resolution of symmetrised tensor products or plethysms[27,28,30-34]. Physical applications required the construction of symmetrised states and in the case of non-compact groups that requires the resolution of symmetrised products of infinite dimensional irreducible representations . Clearly in most cases this will involve incomplete resolutions involving a finite set of irreducible representations . However, a few complete resolutions have been achieved[34]. In particular the second powers of the basic irreducible representations of $Sp(2n,\Re)$ have been fully resolved to give

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{2\} = \sum_{i=0}^{\infty}\langle 1(4i)\rangle \tag{5.1}$$

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{1^2\} = \sum_{i=0}^{\infty}\langle 1(2+4i)\rangle \tag{5.2}$$

$$\langle\tfrac{1}{2}(1)\rangle \otimes \{2\} = \sum_{i=0}^{\infty}\langle 1(2+4i)\rangle \tag{5.3}$$

$$\langle\tfrac{1}{2}(1)\rangle \otimes \{1^2\} = \langle 1(1^2)\rangle + \sum_{i=1}^{\infty}\langle 1(4i)\rangle \tag{5.4}$$

Eq. (5.2) and (5.3) imply

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{1^2\} \equiv \langle\tfrac{1}{2}(1)\rangle \otimes \{2\} \tag{5.5}$$

which implies the $S-$function identity

$$M_+ \otimes \{1^2\} \equiv M_- \otimes \{2\} \tag{5.6}$$

where M_+ and M_- are respectively the infinite $S-$function series involving one-part partitions corresponding to even and odd integers respectively.

Even more remarkable is the result

$$\langle\tfrac{1}{2}(0)\rangle \otimes \{21^2\} \equiv \langle\tfrac{1}{2}(0)\rangle \otimes \{31\} \tag{5.7}$$

Detailed proofs of the above results have been given elsewhere[27,35].

6. Physical implications of the plethysm identities for $Sp(2n,\Re)$

The plethysm identities given by (5.5) and (5.7) imply simple, and seemingly hitherto unnoticed, relationships between particular states for two or four particles in an isotropic three-dimensional harmonic oscillator potential. For a single fermion such as an electron we have the energy level diagram given below.

n		ℓ	$U(3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	———	s,d,g	$\{4\}$
3	———	p,f	$\{3\}$
2	———	s,d	$\{2\}$
1	———	p	$\{1\}$
0	———	s	$\{0\}$

Fig. 1 Energy levels of a single particle in an isotropic three-dimensional harmonic oscillator potential.

For convenience we fix the groundstate energy as zero. Then successive levels have an energy (in appropriate units)

$$E_n = n \tag{6.1}$$

Consider now two non-iteracting particles. Their energies are additive and hence if orbitals with $n = n_1$ and $n = n_2$ are singly occupied then the energy is

$$E_{n_1,n_2} = n_1 + n_2 \tag{6.2}$$

The identity (5.5) then implies that there is a one-to-one mapping between the two-particle spin triplet states ($S = 1$) formed from orbitals of *even* parity with the two-particle spin singlet states ($S = 0$) formed from orbitals of *odd* parity. Thus for $n_1 + n_2 = 2$ we obtain the two sets of states ^3SD and ^1SD while for $n_1 + n_2 = 4$ we obtain the two sets of states $^3SPD_2FG$ and $^1SPD_2FG$ and so on.

In the case of four non-interacting nucleons in an isotropic three-dimensional harmonic oscillator potential the identity observed in (5.7) implies that the the states arising from the left-hand-side of (5.7) are associated with the Wigner isospin-spin $SU(4)$ super-multiplet $\{31\}$ and those of the right-hand-side with the Wigner super-multiplet $\{21^2\}$. The equivalence in (5.7) thus relates the $U(3)$ orbital states involving the occupation of four *even* parity orbitals with those involving the occupation of four *odd* parity orbitals. Thus in the four-nucleon configuration $(0s)^2(1s + 0d)^2$ we have the $U(3) \times SU(4)$ multiplet $\{31\} \times \{31\}$ while in the four-nucleon configuration $(0p)^4$ we have the $U(3) \times SU(4)$ multiplet $\{31\} \times \{21^2\}$.

7. Relationships between irreducible representations

Relationships between the different non-compact groups and their irreducible representations may be established by starting, in the first case, starting with the metaplectic group $Mp(4nk)$ we may relate the decompositions involving the non-compact subgroups $SO^*(2n)$ and $Sp(2n, \Re)$ by means of the commutative diagram[28]

$$\begin{array}{ccccc} SO^*(2n) \times Sp(2k) & \longleftarrow & Mp(4nk) & \longrightarrow & Sp(2n,R) \times O(2k) \\ \downarrow & & & & \downarrow \\ U(n) \times Sp(2k) & & & & U(n) \times O(2k) \\ \downarrow & & & & \downarrow \\ U(n) \times SO(2k) & \longrightarrow & & \longleftarrow & U(n) \times SO(2k) \end{array} \tag{7.1}$$

The terminal group in each case is $U(n)\times SO(2k)$. Taking into account the labels used to distinguish mutually associate pairs of irreducible representations of $Sp(2n\Re)$[27], the decomposition of the metaplectic irreducible representation $\tilde{\Delta}$ of $Mp(4nk)$ proceeds as indicated below:

$$
\begin{tikzcd}
\sum_\kappa [k(\kappa)]\times\langle\kappa\rangle \arrow[d] & \tilde{\Delta} \arrow[l] \arrow[r] & \sum_\lambda \langle k(\lambda)\rangle\times[\lambda] \arrow[d] \\
\sum_\kappa [k(\kappa)]_{U(n)}\times\langle\kappa\rangle \arrow[d] & & \sum_\lambda \langle k(\lambda)\rangle_{U(n)}\times[\lambda] \arrow[d] \\
\sum_\kappa [k(\kappa)]_{U(n)}\times[\kappa/AD] \arrow[r] & {} & \sum_\lambda \big(\langle k(\lambda+(1-\delta_{\lambda_1' k})\lambda^*)\rangle_{U(n)}\times[\lambda] \arrow[l]
\end{tikzcd}
\tag{7.2}
$$

where the symbols $[\cdots]_{U(n)}$ and $\langle\cdots\rangle_{U(n)}$ signify restriction from $SO^*(2n)$ and $Sp(2n,\Re)$, respectively, to $U(n)$, while the skew products of κ with A and D correspond to passing from $Sp(2k)$ up to $U(2k)$ and then down to $SO(2k)$. It should be noted that at the level of $U(n)\times SO(2k)$ the summations over both κ and λ are restricted so that these partitions have no more than P parts with $P=\min(k,n)$.

Since[36]

$$AD = W = \sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^s\{r,s\} \quad \text{with } r-s \text{ even}, \tag{7.3}$$

it follows that on comparing the terms of the form $\cdots\times[\lambda]$ we have

$$[k(\lambda\cdot W)]_{U(n)} = \langle k(\lambda)\rangle_{U(n)} + \big(1-\delta_{\lambda_1' k}\big)\langle k(\lambda^*)\rangle_{U(n)}. \tag{7.4}$$

As special cases of this with $k=1$ and $\lambda=(0)$ and (1), we obtain:

$$(H_+)_{U(n)} = [1(M_+)]_{U(n)} = \langle 1(0)\rangle_{U(n)} + \langle 1(0^*)\rangle_{U(n)}; \tag{7.5a}$$

$$(H_-)_{U(n)} = [1(M_-)]_{U(n)} = \langle 1(1)\rangle_{U(n)}. \tag{7.5b}$$

This gives us an alternative method of computing powers of the basic harmonic representation H of $SO^*(2n)$ and its constituents H_+ and H_-. Since $H=H_++H_-$ it follows from (7.5) that the $U(n)$ content of the harmonic representation H of $SO^*(2n)$ coincides with that of the representation S of $Sp(2n,\Re)$, where

$$S = \langle 1(0)\rangle + \langle 1(0^*)\rangle + \langle 1(1)\rangle. \tag{7.6}$$

The same must be true of both their powers and plethysms. Examples of the application of such a method is given elsewhere[28].

The groups $U(p,q)$ and $Sp(2n,\Re)$ may be similarly related, in this case via the metaplectic group $Mp(2(p+q)k)$ via the commutative diagram below

$$
\begin{tikzcd}
U(p,q)\times U(k) \arrow[d] & Mp(2(p+q)k) \arrow[l] \arrow[r] & Sp(2n,\Re)\times O(k) \arrow[d] \\
U(p,q)\times O(k) \arrow[d] & & U(p+q)\times O(k) \arrow[d] \\
U(p)\times U(q)\times O(k) \arrow[r] & {} & U(p)\times U(q)\times O(k) \arrow[l]
\end{tikzcd}
\tag{7.7}
$$

It is not difficult to see from the diagram that in terms of their $U(p) \times U(q)$ decompositions

$$\langle \tfrac{1}{2}(0)\rangle \sim H_+ \quad \text{and} \quad \langle \tfrac{1}{2}(1)\rangle \sim H_- \tag{7.8}$$

Recalling that

$$\langle \tfrac{1}{2}(0)\rangle \otimes \{1^2\} \equiv \langle \tfrac{1}{2}(1)\rangle \otimes \{2\} \tag{7.9}$$

leading immediately to the non-trivial plethysm identity for $U(p,q)$

$$H_+ \otimes \{1^2\} \equiv H_- \otimes \{2\} \tag{7.10}$$

and likewise that

$$H_+ \otimes \{21^2\} = H_- \otimes \{31\} \tag{7.11}$$

8. $U(2,2)$ plethysms and two-electron systems

As noted earlier, the group $U(2,2)$ can give a description of the states of a one-electron hydrogenic atom. Thus one finds

$$U(2,2) \to U(2) \times U(2) \to SU(2) \times SU(2) \sim SO(4) \tag{8.1a}$$

$$\{1(\bar{0};0)\} \to \sum_{j=0}^{\infty} \{\bar{j}\} \times \{j\} \to \sum_{j=0}^{\infty} \{j\} \times \{j\} \sim \sum_{n=1}^{\infty} [n-1,0] \tag{8.1b}$$

where $Dim[n-1,0] = n^2$ and n may be identified with the principal quantum number of the hydrogenic orbitals as below:-

$n =$	1	2	3	4	$\dots$	
$n^2 =$	1	4	9	16	$\dots$	
$\ell =$	s	s,p	s,p,d	s,p,d,f	$\dots$	(8.2)

the second power of the fundamental irreducible representation $\{1(\bar{0};0)\}$ may be fully resolved as[30]

$$\{1(\bar{0};0)\} \otimes \{2\} = \sum_{k=0}^{\infty} \{2(\overline{2k};2k)\} \tag{8.3a}$$

$$\{1(\bar{0};0)\} \otimes \{1^2\} = \sum_{k=0}^{\infty} \{2(\overline{2k+1};2k+1)\} \tag{8.3b}$$

These plethysms give a full description of the infinite sets of discrete states that are possible for two non-interacting electrons in a central Coulomb field. Eq.(8.3a) is associated with the spin singlet states ($S = 0$ and (8.3b) with the spin triplet states ($S = 1$). One obtains an infinite set of infinite towers of states of well-defined spin. Branching the $U(2,2)$ irreducible representations according to (8.1a) leads to the observation that the groundstate ${}^1S$ arises from the $\{2(\bar{0};0)\}$ irreducible representation as the first state in an infinite tower while the first triplet states ${}^3SP$ arise from the $\{2(\bar{1};1)\}$ irreducible representation as the first triplet states in another infinite tower. At least at this stage we know the complete sets of relevant irreducible representations .

9. Concluding remarks

We can now calculate branching rules, tensor products and plethysms for the harmonic discrete series of representations of the non-compact Lie groups

$Sp(2n, \Re)$, $SO^*(2n)$ and $U(p,q)$. In the process many new insights into properties of irreducible representations have been obtained. A complete understanding of tensor products of finite with infinite representations has yet to be obtained though the preliminary results of Toumazet are encouraging. We have limited our attention to the discrete harmonic series of irreducible representations and clearly future work should consider the continuous irreducible representations and indeed the whole diversity of irreducible representations of these groups. Applications will also require the efficient evaluation of Clebsch-Gordan coefficients and matrix elements[37,38].

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