Plethysm for the non-compact group $Sp(2n, R)$ and new $S-$function identities

K. Grudzinski†§ and B. G. Wybourne†¶

† Instytut Fizyki, Uniwersytet Mikołaja Kopernika, ul. Grudziądzka 5/7, 87-100 Toruń, Poland

Abstract. Methods of computing plethysms of the fundamental unitary irreducible representations of the non-compact symplectic group $Sp(2n, R)$ are considered. Complete results are given for the symmetrised second powers. A number of new $S-$function identities are reported. The stability properties of the $Sp(2n, R)$ plethysms are noted as well as a remarkable conjugacy relation. The application of the plethysms to $N-$particles in an isotropic harmonic oscillator is briefly outlined.

§ E-mail: kagru@phys.uni.torun.pl

¶ E-mail: bgw@phys.uni.torun.pl

1. Introduction

The symplectic group $Sp(6,R)$ is well-known as the dynamical group of the isotropic three-dimensional harmonic oscillator[1]. For a single particle the even-parity states span a single infinite-dimensional irrep commonly denoted[2,3] as $<\frac{1}{2}(0)>$ while the odd-parity states span the irrep $<\frac{1}{2}(1)>$ of $Sp(6,R)$. Collectively they span a single irrep $\tilde{\Delta}$ of the metaplectic group $Mp(6)$, the covering group of $Sp(6,R)$. These groups find significant applications in many-body symplectic models of nuclei[4] and in the mesoscopic properties of quantum dots[5,6]. A central problem in making applications is the resolution of Kronecker powers of the fundamental irreps of $Sp(6,R)$ into their various symmetry types. Basic methods are known[7−9] for computing such resolutions for the powers of the reducible representation $<\frac{1}{2}(0)> + <\frac{1}{2}(1)>$. However, it is desirable to also resolve separately the Kronecker powers of the two fundamental irreps of $Sp(6,R)$ and it this problem we address herein.

Such problems fall in the domain of plethysms[7−11]. We present a method of systematically evaluating plethysms for the fundamental irreps of the group $Sp(6,R)$. In principal the same method applies for all $Sp(2n,R)$. The Kronecker squares of the fundamental irreps for all $Sp(2n,R)$ are fully resolved which in turn leads to a number of new $S-$function identities as well as a new insight into two-particle states. Explicit calculation of the fourth powers leads to a surprising result that implies a remarkable $S-$function identity for certain infinite series of $S-$functions.

We illustrate the application of the method by a brief discussion of the two and three particle states in the symplectic model.

2. The $Sp(2n,R) \to U(n)$ reduction

The general problem of the $Sp(2n,R) \to U(n)$ reduction has been studied in some detail[2,3]. Under that restriction the two fundamental irreps of $Sp(2n,R)$ decompose as[2,3]

$$\begin{aligned} <\frac{1}{2}(0)> &\to \varepsilon^{\frac{1}{2}}(\{0\}+\{2\}+\{4\}+\ldots) \\ &= \varepsilon^{\frac{1}{2}} M_+ \end{aligned} \tag{1}$$

$$\begin{aligned} <\frac{1}{2}(1)> &\to \varepsilon^{\frac{1}{2}}(\{1\}+\{3\}+\{5\}+\ldots) \\ &= \varepsilon^{\frac{1}{2}} M_- \end{aligned} \tag{2}$$

where M_+ and M_- are respectively the *even* and *odd* terms of the infinite $S-$function series indexed by the one part partitions (m) with $m = 0, 1, \ldots, \infty$.

In general

$$<\frac{k}{2}(\lambda)> \to \varepsilon^{\frac{k}{2}} \cdot \{\{\lambda_s\}_N^k \cdot D_N\}_N \tag{3}$$

where $N = min(n,k)$, $\{\lambda_s\}^k$ is a *signed sequence*[2] of terms $\pm\{\rho\}$ such that $\pm[\rho]$ is equivalent to $[\lambda]$ under the modification rules[12−14] of the group $O(k)$, D_N is the infinite $S-$function series indexed by even partitions into not more than N parts. The first $\cdot$ indicates a product in $U(n)$ and the second $\cdot$ a product in $U(N)$ as implied by the final subscript N. Specific examples may be found elsewhere[2,3].

Clearly, Eqs.(1) to (3) will involve an infinite series of irreps of $U(N)$ and any practical calculations must be truncated at some bound. Such calculations can be readily made using the programme SCHUR[15]. The irreps $<\frac{k}{2}(\lambda)>$ of $Sp(2n,R)$ are constrained by the requirement that the conjugate partition $(\tilde{\lambda}) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ satisfy the constraints

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq k \tag{4a}$$

$$\tilde{\lambda}_1 \leq n \tag{4b}$$

The value of $\frac{1}{2}k$ may be an integer or half-odd integer. In that respect it is useful to introduce the equivalent notation

$$< s\kappa;(\lambda) > \equiv < \frac{k}{2}(\lambda) > \tag{5}$$

where

$$\frac{k}{2} = s + \kappa \tag{6}$$

with κ being the integer part of $\frac{k}{2}$ and the residue part is $s = 0$ or $\frac{1}{2}$. Thus the two fundamental irreps will henceforth be designated as $< s;(0) >$ and $< s;(1) >$.

It is critical to our analysis to note that under the reduction $Sp(2n,R) \to U(n)$ the lowest weight $U(n)$ irrep appearing in the decomposition is the irrep $\{\lambda\}$.

3. Evaluation of plethysms for $Sp(2n,R)$

The evaluation of plethysms of the type $(< s;(0) > + < s;(1) >) \otimes \{\nu\}$ has been discussed elsewhere[8,9] using the group chain

$$Sp(2nk,R) \supset Sp(2n,R) \times O(k) \supset Sp(2n,R) \times S(k) \tag{7}$$

with the $O(k) \to S(k)$ decomposition playing a key role. Plethysms in $Sp(2n,R)$ of the type

$$< \frac{k}{2}(\lambda) > \otimes \{\nu\} = \sum_{\tau} c^{\tau}_{\nu} < \frac{\ell}{2}(\tau) > \tag{8}$$

with

$$\ell = k \times |\nu| \tag{9}$$

require a different approach.

Here we proceed by first doing the branching $Sp(2n,R) \rightarrow U(n)$ to give

$$< \frac{k}{2}(\lambda) > \rightarrow \sum_{\rho} c_{\lambda}^{\rho}\{\rho\} \tag{10}$$

where the coefficients c_{λ}^{ρ} are non-negative integers. The sum is infinite with the lowest weight irrep of $U(n)$ being $\{\lambda\}$ with

$$c_{\lambda}^{\lambda} = 1 \tag{11}$$

The next step is to evaluate the plethysms in $U(n)$ to some user chosen cutoff. This gives a list of $U(n)$ irreps which may be ordered in increasing weight starting with the lowest $\{\rho_m\}$. This observation implies that the $Sp(2n,R)$ irrep $< \frac{\ell}{2}(\rho_m) >$ occurs in the $Sp(2n,R)$ plethysm. Thus we may remove from the list of $U(n)$ irreps all those derived from that $Sp(2n,R)$ irrep. The lowest weight irrep of the residue $U(n)$ is identified and the $U(n)$ content of the next $Sp(2n,R)$ irrep removed. This process is continued up to the chosen cutoff.

The above process may be illustrated by calculating the plethysm $< \frac{3}{2}(21) > \otimes\{2\}$ up to terms of maximum weight 10. We first compute the $Sp(6,R) \rightarrow U(3)$ branching rule keeping all terms of weight ≤ 10 to obtain

$$\begin{array}{lllllll}
< \frac{3}{2}(21) > \rightarrow & \{81\} & +\{72\} & +\{71^2\} & +\{63\} & +\{621\} & +\{61\} \\
& +\{54\} & +2\{531\} & +\{52\} & +\{51^2\} & +\{432\} & +\{43\} \\
& +\{421\} & +\{41\} & +\{3^21\} & +\{32\} & +\{31^2\} & +\{21\}
\end{array}$$

We now compute the plethysm at the $U(3)$ level again keeping all terms of weight ≤ 10 to give the following list of $U(3)$ irreps

$$\begin{array}{llllll}
2\{82\} & +\{81^2\} & +3\{73\} & +7\{721\} & +5\{64\} & +11\{631\} \\
+9\{62^2\} & +\{62\} & +\{61^2\} & +\{5^2\} & +10\{541\} & +11\{532\} \\
+2\{53\} & +4\{521\} & +8\{4^22\} & +\{4^2\} & +4\{43^2\} & +4\{431\} \\
+3\{42^2\} & +\{42\} & +2\{3^22\} & +\{321\} & +\{2^3\} &
\end{array}$$

There are three irreps of weight 6 in the above list $(\{42\}, \{321\}, \{2^3\}$ allowing us to immediately conclude that the $Sp(6,R)$ irreps $< 3(42) >, < 3(321) >, < 3(2^3) >$ must occur in the plethysm. These three irreps may be branched to $U(3)$ and the resulting $U(3)$ irreps of weight ≤ 10 removed from the list to leave the $U(3)$ residue

$$\begin{array}{llllll}
\{82\} & +\{81^2\} & +2\{73\} & +5\{721\} & +3\{64\} & +8\{631\} \\
+5\{62^2\} & +\{61^2\} & +\{5^2\} & +7\{541\} & +7\{532\} & +\{53\} \\
+2\{521\} & +4\{4^22\} & +3\{43^2\} & +2\{431\} & +\{3^22\} &
\end{array}$$

Inspection of the above list shows that there are seven irreps of weight eight and hence seven more $Sp(6,R)$ irreps. Continuing we readily find $< \frac{3}{2}(21) > \otimes\{2\}$ contains, to weight 10, the $Sp(6,R)$ irreps

$$
\begin{array}{lllll}
<3;(82)> & + <3;(73)> & +\,2<3;(721)> & +\,2<3;(64)> & +\,2<3;(631)> \\
+\,3<3;(62^2)> & + <3;(61^2)> & +\,2<3;(541)> & + <3;(532)> & + <3;(53)> \\
+\,2<3;(521)> & +\,2<3;(4^22)> & +\,2<3;(431)> & + <3;(42)> & + <3;(3^22)> \\
+ <3;(321)> & + <3;(2^3)> & & &
\end{array}
$$

The plethysms of the irreps $< s;(0) >$ and $< s;(1) >$ are of particular interest in physics applications. The resolution of their Kronecker squares is straightforward. The terms, to weight 16, for plethysms for up to power 4 are relevant to the description of the states of two to four particles in an isotropic three-dimensional harmonic oscillator and have been evaluated. The tabulated results are available at the WEB site http://www.phys.uni.torun.pl/~bgw/.

4. The Kronecker square of the fundamental irreps

Inspection of the symmetrised powers of the irreps $< s;(0) >$ and $< s;(1) >$ reveals a number of surprising features. It would appear that

$$< s;(0) > \otimes\{2\} \;= \sum_{i=0}^{\infty} < 1;(0+4i) > \tag{12}$$

$$< s;(0) > \otimes\{1^2\} = \sum_{i=0}^{\infty} < 1;(2+4i) > \tag{13}$$

$$< s;(1) > \otimes\{2\} \;= \sum_{i=0}^{\infty} < 1;(2+4i) > \tag{14}$$

$$< s;(1) > \otimes\{1^2\} = < 1;(1^2) > + \sum_{i=0}^{\infty} < 1;(4+4i) > \tag{15}$$

holds for all $Sp(2n,R)$ with $n \geq 2$. For $n=1$ the irrep $< 1;(1^2) >$ in Eq.(15) must be deleted. The correctness of Eqs. (12) to (15) may be verified by first noting that the Kronecker squares of the fundamental irreps are $n-$independent for $n \geq 2$ and then using $S-$function identities for the infinite series.

Remarkably, the irrep content in Eqs. (13) and (14) are identical and hence

$$< s;(0) > \otimes\{1^2\} \equiv < s;(1) > \otimes\{2\} \tag{16}$$

which in turn implies a number of hitherto unnoticed identities for plethysms. Even more remarkable is the observation that suggests the conjectured equivalence that

$$< s;(0) > \otimes\{21^2\} \equiv < s;(1) > \otimes\{31\} \tag{17}$$

The equality is evidently $n-$independent for $n \geq 3$. Such an equivalence would only be possible if both plethysms under $Sp(2n,R) \to U(n)$ yielded the same set of $U(n)$ irreps. But this would again require a remarkable $S-$function plethysm identity.

5. Plethysm identities for infinite series of $S-$functions

The equivalence observed in Eq.(16) implies that

$$M_+ \otimes \{1^2\} \equiv M_- \otimes \{2\} \tag{18}$$

Such an equivalence may be readily proved using the properties of the infinite series of $S-$functions defined elsewhere[7,16]. The proof follows by first noting that

$$2M_\pm = M \pm P \tag{19}$$

where

$$M = \sum_{m=0}^{\infty} \quad \text{and} \quad P = \sum_{m=0}^{\infty} \tag{20}$$

and that

$$M_+^2 - M_-^2 = MP = W \tag{21}$$

Then

$$(2M_+) \otimes \{1^2\} = 2(M_+ \otimes \{1^2\}) + M_+^2 = (M + P) \otimes \{1^2\} \tag{22}$$

leading to

$$2(M_+ \otimes \{1^2\}) = (M + P) \otimes \{1^2\} - M_+^2 \tag{23a}$$
$$2(M_- \otimes \{2\}) \;\; = (M - P) \otimes \{2\} - M_-^2 \tag{23b}$$

Thus Eq.(18) will be valid if

$$(M + P) \otimes \{1^2\} - (M - P) \otimes \{2\} = M_+^2 - M_-^2 = W \tag{24}$$

Expanding the left-handside we obtain

$$M \otimes (\{1^2\} - \{2\}) + 2W = -M \otimes p_2 + 2W = W \tag{25}$$

which establishes the conjectured equality. From the equality it follows that

$$M \otimes \{2\} = MM_+ \quad \text{and} \quad M \otimes \{1^2\} = MM_- \tag{26}$$

In precisely the same manner one finds

$$L + \otimes\{1^2\} \equiv L_- \otimes \{2\} \tag{27}$$

where L_+ and L_- are respectively the positive and negative terms of the series

$$L = \sum_{m=0}^{\infty}(-1)^m\{1^m\} \tag{28}$$

Still further identities arise for the infinite $S-$function series defined by

$$A_\pm = \{1^2\} \otimes L_\pm, \;\; B_\pm = \{1^2\} \otimes M_\pm, \;\; C_\pm = \{2\} \otimes L_\pm, \;\; D_\pm = \{2\} \otimes M_\pm \tag{29}$$

Use of the associativity property of plethysms[10] leads directly to

$$Z_+ \otimes \{1^2\} \equiv Z_- \otimes \{2\} \tag{30}$$

for $Z = A, B, C, D$. Furthermore,

$$Z \otimes \{2\} = ZZ_+ \quad \text{and} \quad Z \otimes \{1^2\} = ZZ_- \tag{31}$$

Now to the remarkable Eq. (17). This plethysm implies that

$$M_+ \otimes \{21^2\} \equiv M_- \otimes \{31\} \tag{32}$$

Three independent proofs of this identity have been established. The author first, rather tediously constructed a proof similar to that given for Eq. (18), next Thibon[17] gave a simple proof based upon a power sum expansion of both sides of Eq. (32), finally King[18] used the associativity property of plethysms to give

$$\begin{aligned} &M_+ \otimes \{21^2\} = M_+ \otimes (\{1^2\} \otimes \{1^2\}) \\ &= (M_+ \otimes \{1^2\}) \otimes \{1^2\} = (M_- \otimes \{2\}) \otimes \{1^2\} \\ &= M_- \otimes (\{2\} \otimes \{1^2\}) = M_- \otimes \{31\}. \end{aligned} \tag{33}$$

where use has been made of the fact that $\{1^2\} \otimes \{1^2\} = \{21^2\}$ and $\{2\} \otimes \{1^2\} = \{31\}$. King further notes the generalisation

$$M_+ \otimes (\{1^2\} \otimes \{\sigma\}) = M_- \otimes (\{2\} \otimes \{\sigma\}). \tag{34}$$

Again the identities in Eqs. (33) and (34) can be extended to the series given in Eq. (29). King's generalisation, Eq. (34), can give a useful check on computations of $Sp(2n, R)$ plethysms. For example, choosing $\{\sigma\} \equiv \{2\}$ gives the identity

$$M_+ \otimes (\{2^2\} + \{1^4\}) = M_- \otimes (\{2^2\} + \{4\}) \tag{35}$$

6. Stable $Sp(2n, R)$ plethysms

A given plethysm, Kronecker product or decomposition will be said to be *stable* if at the stable value of $n = n_s$ there is a one-to-one mapping between the resultant list of irreps obtained at the stable value n_s and those obtained for all values of $n > n_s$. Eqs. (1) and (2) are examples of stable decompositions under $Sp(2n, R) \to U(n)$ with a stable value of $n_s = 2$. Likewise the decomposition in Eq. (3) is stable for all values of $n \geq k$.

It follows from King and Wybourne[3] Eq. (8.18) that the $Sp(2n,R)$ Kronecker product

$$< \frac{k}{2}(\lambda) > \times < \frac{\ell}{2}(\nu) > = < \frac{(k+\ell)}{2}((\{\lambda_s\}^k \cdot \{\nu_s\}^\ell \cdot D))_{k+\ell,n} > \tag{36}$$

is certainly stable for all $n \geq (k+\ell)$. We say *certainly* because in certain cases *premature stability* may occur for values of $n < (k+\ell)$. At this point note that all the $S-$functions in Eq. (36) must satisfy, at every stage in the calculation, the constraints of Eqs. (4a) and (4b). This restricts terms in the infinite D series of $S-$functions to those members of the series of length $\ell(\delta) \leq \frac{(k+\ell)}{2}$. Similar restrictions apply to the signed sequences appearing in Eq. (36). As a trivial example consider

$$< 1;(0) >< 1;(0) > = < 2;((\{0_s\}^2 \cdot \{0_s\}^2 \cdot D))_{4,n} >$$

We anticipate stabilisation at $n = 4$ but

$$\{0_s\}^2 = \{0\} - \{2^3\}$$

However, $\{2^3\}$ cannot satisfy the constraints of (8.19) for $n \leq 4$ and should be discarded. Furthermore only the terms of the D series of length 2 can satisfy Eqs. (4a) and (4b) and hence the product stabilises at $n = 2$.

One observes that the third-order plethysms for the two fundamental irreps stabilise at $n = 3$. This is consistent with the stabilisation of the products $< s;(0) >< 1;(\mu) >$ and $< s;(1) >< 1;(\mu) >$ at $n = 3$ and for similar reasons stabilisation of the N-th order plethysms must occur at $n = N$ as observed. Again premature stabilisation for individual plethysms may occur for $n < N$. Thus for $N = 3$ all the plethysms stabilise at $n = 2$ except for $< s;(1) > \otimes\{1^3\}$ which stabilises at $n = 3$. Stabilisation for arbitrary N stabilisation occurs at $n = N-1$ except for $< s;(1) > \otimes\{1^N\}$ which stabilises at $n = N$.

7. Conjugacy mappings

Inspection of tables for the plethyms $< s;(0) > \otimes\{\lambda\}$ and $< s;(1) > \otimes\{\tilde{\lambda}\}$ where $\tilde{\lambda}$ is the conjugate of λ suggests that the two plethysms are remarkably related by one-to-one mappings such that if

$$< s;(0) > \otimes\{\lambda\} = \sum_{\mu} g^{\mu} < k;(\mu) > \tag{37}$$

where $k = |\lambda|/2$ and g^{μ} is the multiplicity, then the terms $g^{\mu} < k;(\mu) >$ in $< s;(1) > \otimes\{\tilde{\lambda}\}$ are identical to those in Eq. (37) apart from those that are related by the following simple (μ) one-to-one mappings

$$\begin{array}{llllll}
\lambda \vdash 2 & (0) \to (1^2) & & & & \\
\lambda \vdash 3 & (0) \to (1^3) & (a) \to (a1) & (a1) \to (a) & & \\
\lambda \vdash 4 & (0) \to (1^4) & (a) \to (a1^2) & (a1^2) \to (a) & & \\
\lambda \vdash 5 & (0) \to (1^5) & (a) \to (a1^3) & (a1^3) \to (a) & (ab) \to (ab1) & (ab1) \to (ab) \\
\lambda \vdash 6 & (0) \to (1^6) & (a) \to (a1^4) & (a1^4) \to (a) & (ab) \to (ab1^2) & (ab1^2) \to (ab)
\end{array} \tag{38}$$

That such simple relationships seem to exist is by no means evident from the methods used to establish the plethysms and hints at an underlying simplicity that remains to be discovered and a conjugacy theorem still to be exposed.

8. Two-particle states

The plethysm equivalence noted in Eq. (16) has consequences for the case of the states of two non-interacting fermions in an isotropic three-dimensional harmonic oscillator potential. It means that for the even-parity two-particle states there is a one-to-one correspondence between the spin triplet states formed by two-particles in even-parity orbitals with the spin singlet states formed by two particles in odd-parity orbitals, a feature of the much studied isotropic three-dimensional harmonic oscillator potential that does not seem to have been hitherto observed.

The corresponding plethysm equivalence noted in Eq. (17) is less applicable since for N spin $\frac{1}{2}$ identical fermions the Pauli exclusion principle excludes spin states involving irreps of of $S(N)$ involving partitions into more than two parts. In the case of nucleons where spin and isospin are considered irreps of $S(N)$ involving partitions into up to four parts arise and some application is possible but not in the form found so directly for two-particles.

9. Three-particle states

There is no difficulty, in principle, in determining the states for $N-$particles in an isotropic three-dimensional harmonic oscillator. The case of three particles suffices to illustrate the general procedure. For three particles in an isotropic three-dimensional harmonic oscillator potential the dynamical group is $Mp(18)$ whose fundamental irrep $\tilde{\Delta}$ decomposes under restriction to $Sp(18,R)$ as

$$\tilde{\Delta} \to < \frac{1}{2}(0) > + < \frac{1}{2}(1) > \tag{39}$$

Then under $Sp(18,R) \to Sp(6,R) \times O(3)$

$$\begin{aligned}
< \tfrac{1}{2}(0) > \to \quad & < s1;(0) > [0] + < s1;(1^2) > [1]\# + < s1;(2) > [2] \\
& + < s1;(31) > [3]\# + < s1;(4) > [4] + < s1;(51) > [5]\# \\
& + < s1;(6) > [6] + < s1;(71) > [7]\# + < s1;(8) > [8] \\
& + < s1;(91) > [9]\# + < s1;(10\ ) > [10\ ] \\
< \tfrac{1}{2}(1) > \to \quad & < s1;(1) > [1] + < s1;(1^3) > [0]\# + < s1;(21) > [2]\# \\
& + < s1;(3) > [3] + < s1;(41) > [4]\# + < s1;(5) > [5] \\
& + < s1;(61) > [6]\# + < s1;(7) > [7] + < s1;(81) > [8]\# \\
& + < s1;(9) > [9] + < s1;(10\ 1) > [10\ ]\#
\end{aligned} \tag{40}$$

The spins associated with these representations can be found from a knowledge of the $O(3) \to S(3)$ branching rules. Note that to obtain the branching rule for $[n]\#$ one simply replaces the $S(3)$ irreps by their conjugates.

The terms associated with the $\{3\}$ irrep of $S(3)$ are spurious while those with $\{21\}$ and $\{1^3\}$ correspond to states with spin $S = \frac{1}{2}$ and $\frac{3}{2}$ respectively.

The three-particle states can be equivalently found from the use of the $Sp(6, R)$ plethysms. The *even* parity states must arise from

$$(S = \frac{1}{2}) < \frac{1}{2}(0) > \otimes\{21\} + < \frac{1}{2}(1) > \otimes\{2\} < \frac{1}{2}(0) > + < \frac{1}{2}(1) > \otimes\{1^2\} < \frac{1}{2}(0) > \quad (41)$$

$$(S = \frac{3}{2}) < \frac{1}{2}(0) > \otimes\{1^3\} + < \frac{1}{2}(1) > \otimes\{1^2\} < \frac{1}{2}(0) > \quad (42)$$

while for the *odd* parity states they arise from

$$(S = \frac{1}{2}) < \frac{1}{2}(1) > \otimes\{21\} + < \frac{1}{2}(0) > \otimes\{2\} < \frac{1}{2}(1) > + < \frac{1}{2}(1) >< \frac{1}{2}(0) > \otimes\{1^2\} \quad (43)$$

$$(S = \frac{3}{2}) < \frac{1}{2}(1) > \otimes\{1^3\} + < \frac{1}{2}(0) \otimes \{1^2\} < \frac{1}{2}(1) > \quad (44)$$

To weight 10 we obtain the following even parity states

$$\begin{array}{ll}
(S = \frac{1}{2}) & < s1; (1^2) > + 2 < s1; (2) > + 2 < s1; (31) > + 3 < s1; (4) > \\
 & + 4 < s1; (51) > + 4 < s1; (6) > + 5 < s1; (71) > + 6 < s1; (8) > \\
 & + 6 < s1; (91) > + 7 < s1; (10\ ) > \\
(S = \frac{3}{2}) & < s1; (1^2) > + 2 < s1; (31) > + < s1; (4) > + 2 < s1; (51) > \\
 & + 2 < s1; (6) > + 3 < s1; (71) > + 2 < s1; (8) > + 4 < s1; (91) > \\
 & + 3 < s1; (10\ ) >
\end{array}$$

while for the odd parity states we obtain

$$\begin{array}{ll}
(S = \frac{1}{2}) & < s1; (1) > + 2 < s1; (21) > + 2 < s1; (3) > + 3 < s1; (41) > \\
 & + 4 < s1; (5) > + 4 < s1; (61) > + 5 < s1; (7) > + 6 < s1; (81) > \\
 & + 6 < s1; (9) > \\
(S = \frac{3}{2}) & < s1; (1^3) > + < s1; (21) > + < s1; (3) > + 2 < s1; (41) > \\
 & + < s1; (5) > + 3 < s1; (61) > + 2 < s1; (7) > + 3 < s1; (81) > \\
 & + 3 < s1; (9) >
\end{array}$$

10. Lowest energy states for non-interacting particles

In the case of N non-interacting particles in a harmonic oscillator potential the energy of a given state is simply the sum of the one-particle energies and hence the lowest energy state associated with a given $Sp(6,R)$ multiplet $<\kappa(\lambda)>$ is, relative to the groundstate energy,

$$w_\lambda \hbar\omega \tag{45}$$

where ω is the oscillator angular frequency and w_λ is the weight of the partition (λ). Representations of $Sp(6,R)$ having different partitions but of the same weight will have the same zero-order energy as given in Eq. (45).

For three-particles we have, to weight 6, the $U(3)$ states with spin $S=\frac{1}{2}$ are illustrated in Fig. 1 and those for $S=\frac{3}{2}$ in Fig. 2.

The $U(3)$ states of weight w for $N-$particles may be determined as follows

1. Partition the integer w into N parts allowing zero parts if necessary.
2. Even weight partitions involve even parity states otherwise odd parity states.
3. Replace each part ,i ,by $\{i\}$ which then labels the $U(3)$ irrep for a single particle in the $i-$th harmonic oscillator orbital. A given orbital i can accommodate up to $4i+2$ particles with spin $\frac{1}{2}$.
4. For a given partition containing k distinct non-repeating parts form the $SU(2)\times U(3)$ Kronecker product

$$\{\frac{1}{2}\}\times\{i_1\}.\{\frac{1}{2}\}\times\{i_2\}\cdots.\{\frac{1}{2}\}\times\{i_k\} \tag{46}$$

to give a series of $SU(2)^S\times U(3)$ multiplets.

5. If the parts i are repeated with a multiplicity m then evaluate the plethysm

$$\begin{aligned}\{\frac{1}{2}\}\{i\})\otimes\{1^m\} &= \{\frac{m}{2}\}\{i\}\otimes\{1^m\} \quad if\ m>2\\ &= \{1\}(\{i\}\otimes\{1^2\}+\{0\}(\{i\}\otimes\{2\} \quad if\ m=2\end{aligned} \tag{47}$$

For $N=3$ we have for weight 4 the four partitions

$$4+0+0,\quad 3+1+0,\quad 2+2+0,\quad 2+1+1 \tag{48}$$

Applying the above algorithm we find for the first partition a $U(3)$ multiplet with $S=\frac{1}{2}$. The second partition gives two $U(3)$ multiplets, $\{4\}+\{31\}$ with spins $S=\frac{1}{2}$ and $S=\frac{3}{2}$. The third partition yields the $U(3)$ multiplet $\{31\}$ with $S=\frac{3}{2}$ and the $U(3)$ multiplets $\{4\}+\{31\}+\{2^2\}$ with spin $S=\frac{1}{2}$. The fourth partition yields the two $U(3)$ multiplets $\{31\}+\{2^2\}$ with spin $S=\frac{3}{2}$ and the three $U(3)$ multiplets $\{4\}+2\{31\}+\{2^2\}+\{21^2\}$ with spin $S=\frac{1}{2}$. Thus for spin $S=\frac{3}{2}$ we obtain the $U(3)$ multiplets $\{4\}+3\{31\}+\{21^2\}$ and for spin $S=\frac{1}{2}$ the $U(3)$ multiplets $4\{4\}+4\{31\}+2\{2^2\}+\{21^2\}$ in agreement with those found in Figs. (1) and (2) using the group $Sp(6,R)\Rightarrow U(3)$ decompositions.

11. Concluding remarks

Some basic methods of computing plethysms for the non-compact group $Sp(2n,R)$ have been outlined. A novel, and unexpected, feature of this work has been the recognition of a number of new identities concerning plethysms of the fundamental irreps of $Sp(2n,R)$ and consequential identities involving plethysms of certain infinite series of $S-$functions. These identities give rise to an apparently hitherto unrecognised property of two-particle states in an isotropic harmonic potential.

An initially surprising feature is the essentially $n-$independence of the $Sp(2n,R)$ plethysms. These stabilise for sufficiently large n and results for smaller n follow by rejection from the stabilised result of all $Sp(2n,R)$ irreps that do not satisfy Eq. (4) for the smaller value of n. Increasing n beyond its stabilsed value involves considerably more computation but no new types of $Sp(2n,R)$ irreps.

Much of the preceding work is relevant to symplectic models of nuclei and mesoscopic systems such as quantum dots. It should be possible to start to consider the properties of model Hamiltonians constructed from the group generators. The first step is the determination of the states of the non-interacting particles which has been one of the objectives of this paper.

Acknowledgments

This work was supported by Polish KBN Grant 18/p3/94/07 with some of the work being done at the Max-Planck-Institüt für Astrophysik at Garching. We are particularly appreciative of helpful communications from R C King and J-Y Thibon.

References

[1] Wybourne B G 1974 *Classical Groups for Physicists*(New York: Wiley)
[2] Rowe D J, Wybourne B G and Butler P H 1985 *J. Phys. A:Math. Gen.* **18** 939
[3] King R C and Wybourne B G 1985 *J. Phys. A:Math. Gen.* **18** 3113
[4] Rowe D J 1985 *Rep. Prog. Phys.* **48** 1419
[5] Haase R W and Johnson N F 1993 *Phys. Rev.* **B48** 1583
[6] Wybourne B G 1994 *Rept. Math. Phys.* **34** 9
[7] Yang M and Wybourne B G 1986 *J. Phys. A:Math. Gen.* **19** 3513
[8] Carvalho M J 1990 *J. Phys. A:Math. Gen.* **23** 1909
[9] Wybourne B G 1992 *J. Phys. A:Math. Gen.* **25** 4389
[10] Littlewood D E 1940 *The Theory of Group Characters* (Oxford: Clarendon)
[11] Macdonald I G 1979 *Symmetric functions and Hall Polynomials* (Oxford: Clarendon)
[12] King R C 1971 *J. Math. Phys.***12** 1588
[13] King R C 1975 *J. Phys. A: Math. Gen.***8** 429
[14] Black G R E, King R C and Wybourne B G 1983 *J. Phys. A: Math. Gen.***16** 1555
[15] **SCHUR** An interactive program for calculating properties of Lie groups and symmetric functions, distributed by S. Christensen Email: steve@scm.vnet.net, http://scm.vnet.net/Christensen.html
[16] King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.***14** 2509
[17] J-Y Thibon 1996 (private communication)
[18] King R C 1996 (private communication)

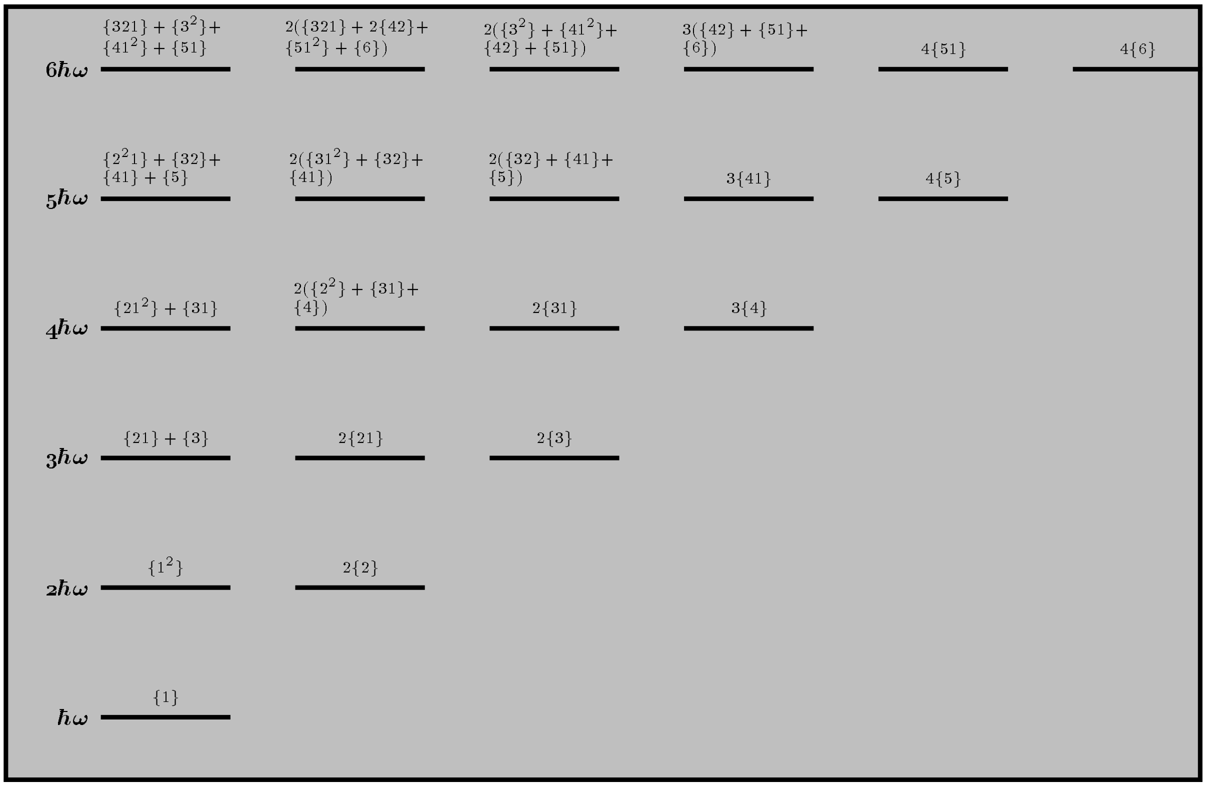

Figure 1: $U(3)$ multiplets to weight 6 for spin $S = \frac{1}{2}$ 3-particle harmonic oscillator states.

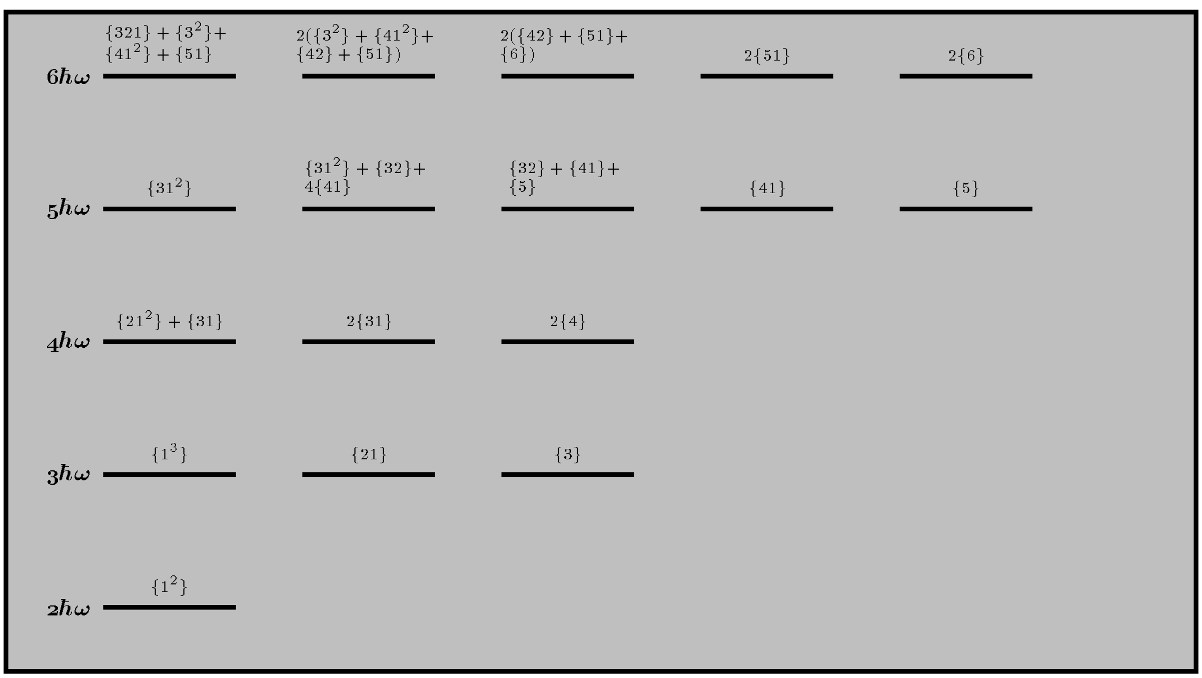

Figure 2: $U(3)$ multiplets to weight 6 for spin $S = \frac{3}{2}$ 3-particle harmonic oscillator states.