Products and symmetrised powers of irreducible representations of $Sp(2n, \Re)$ and their associates

R C King†§ and B. G. Wybourne†¶

† Mathematics Department, University of Southampton, Southampton, England
† Instytut Fizyki, Uniwersytet Mikołaja Kopernika, ul. Grudziądzka 5/7, 87-100 Toruń, Poland

21 May 1998

Abstract. The calculation of Kronecker products and plethysms of the infinite-dimensional harmonic series unitary irreducible representations of the non-compact group $Sp(2n, \Re)$ is considered. The complementarity of $Sp(2n, \Re)$ and $O(k)$ is used to define associate irreducible representations of $Sp(2n, \Re)$. This leads to simple relationships between Kronecker products and plethysms of irreducible representations of $Sp(2n, \Re)$ and those of their corresponding associate irreducible representations. In the process of proving the validity of these previously conjectured relationships several new identities are found for plethysms involving infinite series of Schur functions. In addition, a general formula for plethysms of arbitrary irreducible representations of $Sp(2n, \Re)$ is derived and its implementation is illustrated with a detailed example. A remarkable analogy is then observed between plethysms of the basic harmonic irreducible representations of $Sp(2n, \Re)$ and those of the basic spin irreducible representations of $SO(2n)$.

1. Introduction

The symplectic group $Sp(6, \Re)$ is well-known as the dynamical group for a single particle in an isotropic three-dimensional harmonic oscillator potential[1]. For N-noninteracting particles in an isotropic three-dimensional harmonic oscillator potential the group of interest[2−6] is $Sp(6N, \Re)$. In general the group $Sp(2n, \Re)$ is of relevance to symplectic models of nuclei[4] and certain mesoscopic systems such as quantum dots[5,6]. The irreducible representations of $Sp(2n, \Re)$ of interest in these problems are the infinite-dimensional harmonic series unitary irreducible representations[7]. Methods of calculating their tensor or Kronecker products in terms of infinite series of Schur functions[8,9] (S-functions) have been developed earlier[2,3]. The corresponding problem of resolving symmetrised powers or plethysms of the irreducible representations has also been tackled through the use of infinite series of Schur functions[10−15]. It has been observed that explicit calculations[16] of such plethysms seemed to imply some hitherto unnoticed conjugacy relationships[14,15]. The wish to prove these conjugacy relationships was the principal motivation for developing the content of this paper. Central to their derivation is the use of the complementarity of $Sp(2n, \Re)$ and $O(k)$ which is used to define associate irreducible representations of $Sp(2n, \Re)$. It is this

§ E-mail: rck@maths.soton.ac.uk
¶ E-mail: bgw@phys.uni.torun.pl

that leads to the required conjugacy relationships between both Kronecker products and plethysms of irreducible representations and their associates in $Sp(2n,\Re)$. In the process of proving the most general possible form of these conjugacy relationships it has been necessary to establish a number of new identities and lemmas relating to plethysms involving infinite series of Schur functions. In addition a general formula for the evaluation of plethysms of arbitrary harmonic series irreducible representations of $Sp(2n,\Re)$ is derived and illustrated with a detailed example. Finally detailed consideration is given to the very striking analogy between basic spin irreducible representations of $SO(2n)$ and the basic harmonic irreducible representations of $Sp(2n,\Re)$. This leads to a simplification of earlier analyses[17] of the symmetrised squares and cubes of the basic harmonic irreducible representations of $Sp(2n,\Re)$. The results obtained in this paper represent a further step towards the practical implementation of symplectic models of many-particle systems.

2. Harmonic series unitary irreducible representations of $Sp(2n,\Re)$

Following the terminology and notation of an earlier paper[3] the harmonic series unitary irreducible representations[7] of $Sp(2n,\Re)$ are specified by symbols $\langle\frac{1}{2}k(\lambda)\rangle$ where $\lambda=(\lambda_1,\lambda_2,\ldots)$ is a partition for which the conjugate partition $\lambda'=(\lambda'_1,\lambda'_2,\ldots)$ is such that $\lambda'_1+\lambda'_2\leq k$ and $\lambda'_1\leq n$. The relationship between a partition and its conjugate is such that the parts of λ and λ' specify the row and column lengths, respectively, of the corresponding Young diagram F^λ. If λ is a partition of m then the total number of boxes in F^λ is m, which is sometimes referred to as the *weight* of λ. By the same token the number of boxes, λ'_1, in the first column of F^λ and the number, λ_1, in the first row are referred to as the *length* and *width*, respectively, of λ.

The two basic harmonic series irreducible representations may be denoted by $\tilde{\Delta}_+=\langle\frac{1}{2}(0)\rangle$ and $\tilde{\Delta}_-=\langle\frac{1}{2}(1)\rangle$. Their direct sum

$$\tilde{\Delta}=\tilde{\Delta}_+ +\tilde{\Delta}_- =\langle\tfrac{1}{2}(0)\rangle+\langle\tfrac{1}{2}(1)\rangle \tag{2.1}$$

is the restriction to $Sp(2n,\Re)$ of the defining irreducible representation of the metaplectic group $Mp(2n,\Re)$, the two-sheeted covering group of the symplectic group $Sp(2n,\Re)$. As a representation of $Sp(2n,\Re)$ the basic harmonic or metaplectic representation $\tilde{\Delta}$ is an example of the unitary ray representations introduced for all Lie groups by Bargmann[18]. More precisely it is the infinite-dimensional double-valued projective representation of $Sp(2n,\Re)$ studied in the mathematics literature by Segal[19], Shale[20] and Weil[21], and independently in the physics literature by Moshinsky and Quesne[22]. The connection with the metaplectic group $Mp(2n,\Re)$ is made by Weil[21], while both Shale[20] and Moshinsky and Quesne[22] point out that the metaplectic representation $\tilde{\Delta}$ is the analogue for $Sp(2n,\Re)$ of the basic spin representation of $O(2n)$. The wider class of harmonic series irreducible representations studied here were first introduced by Kashiwara and Vergne[7] as new unitary representations of the metaplectic group $Mp(2n,\Re)$ arising as irreducible components of tensor powers of $\tilde{\Delta}$.

It is convenient to gather together some known facts about these harmonic series irreducible representations: their behaviour on restriction from $Sp(2n,\Re)$ to the maximal compact subgroup $U(n)$; the decomposition of their tensor products; the relationship between their symmetrised products and the branching rule for the restriction of $O(k)$ to the symmetric group S_k.

All of these facts can be deduced by exploiting the fact that the pair of groups $Sp(2n,\Re)$ and $O(k)$ are a dual pair with respect to $Mp(2nk,\Re)$ in the sense of Howe[23] or, equivalently, a complementary pair of subgroups of $Sp(2nk,\Re)$ in the sense of Moshinsky and Quesne[22]. This duality or complementarity is such that on restriction from $Sp(2nk,\Re)$ to $Sp(2n,\Re)\times O(k)$ we have the branching rule:

$$\tilde{\Delta} \to \sum_{\lambda} \langle \tfrac{1}{2}k(\lambda)\rangle \times [\lambda], \tag{2.2}$$

where the summation is over all those λ such that

$$\lambda_1' + \lambda_2' \leq k \quad \text{and} \quad \lambda_1' \leq n. \tag{2.3}$$

Under restriction from $Sp(2n,\Re)$ to its maximal compact subgroup $U(n)$ we have[2,3]

$$\langle \tfrac{1}{2}k(\lambda)\rangle \to \sum_{\mu} \varepsilon^{k/2} R^{\mu}_{\lambda}\ \{\mu\}, \tag{2.4}$$

where the summation is over all those μ such that

$$\mu_1' \leq \ \min(k,n) \tag{2.5}$$

and $\varepsilon = \{1^n\}$ is the 1-dimensional irreducible representation of $U(n)$ in which each group element is mapped to its determinant. The coefficients R^{μ}_{λ} are defined by the branching rule for the restriction from $U(k)$ to $O(k)$:

$$\{\mu\} \to \sum_{\lambda} R^{\mu}_{\lambda}\ [\lambda]. \tag{2.6}$$

The particular significance of (2.4) is not just that it defines the decomposition of the restriction of the irreducible representation $\langle \frac{1}{2}k(\lambda)\rangle$ of $Sp(2n,\Re)$ into irreducible representations of $U(n)$, but that it serves to define completely the character of $\langle \frac{1}{2}k(\lambda)\rangle$ since $Sp(2n,\Re)$ and $U(n)$ are of the same rank, n. Furthermore since every harmonic series representation obtained by taking some arbitrary linear combination of products of the unitary irreducible representations $\langle \frac{1}{2}k(\lambda)\rangle$ is itself unitary, it is fully reducible and its irreducible content is completely determined by its character. Since this may be evaluated at the level of $U(n)$, as on the right-hand side of (2.4), identities between characters at the level of $U(n)$ imply corresponding identities, up to equivalence, between representations at the level of $Sp(2n,\Re)$. This is exploited in what follows.

In order to evaluate explicitly the branching rule coefficients in (2.6) it is convenient to note that it can be expressed in the form[24]

$$\{\mu\} \to [\mu/D] \tag{2.7}$$

where

$$D = \sum_{\delta} \{\delta\} = \{0\} + \{2\} + \{4\} + \{2^2\} + \cdots, \tag{2.8}$$

in which the summation is over all partitions δ having just even parts, and / signifies an S-function quotient. This can then be used[2,3] to rewrite the branching rule (2.4) in the form

$$\langle \tfrac{1}{2}k(\lambda)\rangle \to \varepsilon^{k/2} \cdot \{\lambda_s\}^k \cdot D, \tag{2.9}$$

where $\{\lambda_s\}^k$ is the signed sequence[2,3]

$$\{\lambda_s\}^k = \sum_{\mu} \eta_\mu^\lambda \ \{\mu\}, \tag{2.10}$$

with the summation extending over all μ with $\mu_1' \leq k$ such that $[\mu] = \eta_\mu^\lambda[\lambda]$ under the modification rules[24] of $O(k)$. The non-vanishing coefficients η_μ^λ are all ± 1. The symbol $\cdot$ in (2.8) signifies an S-function product corresponding precisely to a tensor or Kronecker product in $U(n)$. For given n it is only necessary to retain those terms $\{\nu\}$ in the products (2.8) for which $\nu_1' \leq n$.

It should be noted that in the case $k = 1$ the restriction from $Sp(2n, \Re)$ to $U(n)$ is such that the basic harmonic irreducible representations decompose in accordance with the rules:

$$\tilde{\Delta}_+ = \langle \tfrac{1}{2}(0) \rangle \to \varepsilon^{1/2} M_+; \tag{2.11a}$$
$$\tilde{\Delta}_- = \langle \tfrac{1}{2}(1) \rangle \to \varepsilon^{1/2} M_-, \tag{2.11b}$$

where

$$M_+ = \sum_{m:m \text{ even}} \{m\} = \{0\} + \{2\} + \{4\} + \cdots; \tag{2.12a}$$
$$M_- = \sum_{m:m \text{ odd}} \{m\} = \{1\} + \{3\} + \{5\} + \cdots. \tag{2.12b}$$

It has been shown[2] that the tensor product of a pair of unitary harmonic series irreducible representations of $Sp(2n, \Re)$ decomposes in accordance with the rule

$$\langle \tfrac{1}{2}k(\mu) \rangle \times \langle \tfrac{1}{2}\ell(\nu) \rangle = \sum_{\lambda} \ K_\lambda^{\mu\nu} \langle \tfrac{1}{2}(k+\ell)(\lambda) \rangle, \tag{2.13}$$

where the coefficients $K_\lambda^{\mu\nu}$ are the branching rule coefficients appropriate to the restriction $O(k+\ell) \to O(k) \times O(\ell)$:

$$[\lambda] \to \sum_{\mu\nu} \ K_\lambda^{\mu\nu} \ [\mu] \times [\nu]. \tag{2.14}$$

In general it is not so straightforward to decompose symmetrised powers or plethysms of irreducible representations of $Sp(2n, \Re)$. Let ρ be a partition of k. Then in the case of the metaplectic representation $\tilde{\Delta}$, its corresponding k-fold symmetrised power decomposes in accordance with the rule[10,11]:

$$\tilde{\Delta} \otimes \{\rho\} = \sum_{\lambda} \ b_\rho^\lambda \ \langle \tfrac{1}{2}k(\lambda) \rangle, \tag{2.15}$$

where the coefficients b_ρ^λ are the branching rule coefficients appropriate to the restriction $O(k) \to S_k$:

$$[\lambda] \to \sum_{\rho} \ b_\rho^\lambda \ (\rho), \tag{2.16}$$

where the summation is carried out over all partitions ρ of k. The coefficients b^λ_ρ may be found by noting that[25]

$$[\lambda] \rightarrow (k-1,1) \otimes \{\lambda/G\}, \tag{2.17}$$

where

$$\begin{aligned} G &= \sum_\epsilon \, (-1)^{(e-r)/2} \, \{\varepsilon\} \\ &= \{0\} + \{1\} - \{21\} - \{2^2\} + \{31^2\} + \{321\} - \cdots, \end{aligned} \tag{2.18}$$

in which the summation is over all self-conjugate partitions ϵ with e equal to the weight of ϵ and r equal to its Frobenius rank, that is the number of boxes on the main diagonal of the corresponding Young diagram F^ε

3. Associate irreducible representations of $Sp(2n,\Re)$

It is well known[8,24] that corresponding to each irreducible representation $[\lambda]$ of the full orthogonal group $O(k)$ there exists an associate irreducible representation $[\lambda]^*$. The relationship between these irreducible representations is such that if $[\lambda]$: $A \mapsto [\lambda](A)$ for each group element A of $O(k)$, then $[\lambda]^*$:$A \mapsto [\lambda]^*(A) = \det A \cdot [\lambda](A)$. Since $\det A = \pm 1$ for all $A \in O(k)$ it follows that $([\lambda]^*)^* = [\lambda]$.

In terms of the partitions used to label irreducible representations of $O(k)$, if the partition λ labelling $[\lambda]$ has conjugate $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3, \ldots)$ then the partition λ^* labelling $[\lambda]^*$, which is referred to as the k-associate of λ, has conjugate $\lambda^{*\prime} = (k–\lambda'_1, \lambda'_2, \lambda'_3, \ldots)$. Equivalently, the k-associate λ^* of the partition λ is defined by the Young diagram F^{λ^*} obtained from the Young diagram F^λ by taking the complement of the first column with respect to a column of length k.

It should be noted that for each irreducible representation $[\lambda]$ of $O(k)$ the corresponding partition λ is $O(k)$-standard in the sense that $\lambda'_1 + \lambda'_2 \leq k$. This is precisely what is required to guarantee that $k - \lambda'_1 \geq \lambda'_2$ so that λ^* is a partition. Similarly the fact that λ is a partition guarantees that $\lambda'_1 \geq \lambda'_2$ so that ${\lambda^*}'_1 + {\lambda^*}'_2 = k - \lambda'_1 + \lambda'_2 \leq k$. Thus λ^* is also $O(k)$-standard.

As a special case of the above it should be noted that the associate of the identity irreducible representation $[0]$ is just the irreducible representation $[0]^* = [1^k]$ in which each group element A of $O(k)$ is mapped to its determinant. More generally

$$[\lambda]^* = [\lambda^*] = [\lambda] \cdot [0]^* = [\lambda] \cdot [1^k]. \tag{3.1}$$

Returning to $Sp(2n,\Re)$, it is natural thanks to (2.2) to associate with each irreducible representation $\langle \frac{1}{2}k(\lambda)\rangle$ of $Sp(2n,\Re)$ an associate irreducible representation $\langle \frac{1}{2}k(\lambda)\rangle^*$. The complementarity between $Sp(2n,\Re)$ and $O(k)$ embodied in (2.2) then leads to:

Definition 3.1 *For all $k \leq n$ the associate $\langle \frac{1}{2}k(\lambda)\rangle^*$ of the irreducible representation $\langle \frac{1}{2}k(\lambda)\rangle$ of $Sp(2n,\Re)$ is defined by*

$$\langle \tfrac{1}{2}k(\lambda)\rangle^* = \langle \tfrac{1}{2}k(\lambda^*)\rangle \tag{3.2}$$

where λ^* is the k-associate of λ.

As a special case of this with $k = 1$ it is clear that

$$\big(\tilde{\Delta}_+\big)^* = \langle\tfrac{1}{2}(0)\rangle^* = \langle\tfrac{1}{2}(1)\rangle = \tilde{\Delta}_-; \tag{3.3a}$$
$$\big(\tilde{\Delta}_-\big)^* = \langle\tfrac{1}{2}(1)\rangle^* = \langle\tfrac{1}{2}(0)\rangle = \tilde{\Delta}_+. \tag{3.3b}$$

With this notation and terminology it is not difficult to establish the following:
Proposition 3.2 *If under the restriction from $Sp(2n,\Re) \to U(n)$ each irreducible representation $\langle\frac{1}{2}k(\lambda)\rangle$ decomposes in such a way that*

$$\langle\tfrac{1}{2}k(\lambda)\rangle \to \sum_{\mu} \varepsilon^{k/2} R^{\mu}_{\lambda}\{\mu\}, \tag{3.3}$$

then for $k \le n$ the associate irreducible representation $\langle\frac{1}{2}k(\lambda)^*\rangle$ decomposes in accordance with the rule:

$$\langle\tfrac{1}{2}k(\lambda)\rangle^* \to \sum_{\mu} \varepsilon^{k/2} R^{\mu}_{\lambda}\{\mu\}/\{1^k\}, \tag{3.4}$$

where / signifies an S-function quotient.
Proof In (3.3) it should be noted that R^{μ}_{λ} is defined by the $U(k) \to O(k)$ branching rule (2.6). However, in $U(k)$ for all ν such that $\nu'_1 \le k$ we have $\{\nu\}\cdot\{1^k\} = \{\mu\}$ with $\mu'_1 = k$ where F^{μ} is obtained from F^{ν} by adding a leftmost column of length k. Under the restriction from $U(k)$ to $O(k)$ we have

$$\{\mu\} = \{\nu\}\cdot\{1^k\} \to \sum_{\kappa} R^{\nu}_{\kappa}\,[\kappa]\cdot[1^k] = \sum_{\kappa} R^{\nu}_{\kappa}\,[\kappa^*] = \sum_{\lambda} R^{\nu}_{\lambda^*}\,[\lambda]. \tag{3.5}$$

It then follows by comparison with (2.6) that

$$R^{\nu}_{\lambda^*} = R^{\mu}_{\lambda} \quad \text{where} \quad \{\mu\} = \{\nu\}\cdot\{1^k\} \quad \text{and} \quad \{\nu\} = \{\mu\}/\{1^k\}. \tag{3.6}$$

Hence under restriction from $Sp(2n,\Re)$ to $U(n)$, provided that $n \ge k$, we have

$$\begin{aligned}\langle\tfrac{1}{2}k(\lambda)\rangle^* = \langle\tfrac{1}{2}k(\lambda^*)\rangle \to \sum_{\nu:\nu'_1\le k} \varepsilon^{k/2} R^{\nu}_{\lambda^*}\{\nu\} &= \sum_{\mu:\mu'_1=k} \varepsilon^{k/2} R^{\mu}_{\lambda}\{\mu\}/\{1^k\} \\ &= \sum_{\mu:\mu'_1\le k} \varepsilon^{k/2} R^{\mu}_{\lambda}\{\mu\}/\{1^k\}\end{aligned} \tag{3.7}$$

where the last step follows from the fact that $\{\mu\}/\{1^k\} = 0$ if $\mu'_1 < k$. This completes the proof.

The consistency of Proposition 3.2 with what we know of the branching rules (2.10) of the basic harmonic irreducible representations is easy to verify. In this case we have $k = 1$ and as we have seen $\tilde{\Delta}_{\pm} \to \varepsilon^{1/2}M_{\pm}$. It then follows from Proposition 3.2 that $\big(\tilde{\Delta}_{\pm}\big)^* \to \varepsilon^{1/2}M_{\pm}/\{1\} = \varepsilon^{1/2}M_{\mp}$ as can be seen from (2.11) since $\{m\}/\{1\} = \{m-1\}$ for $m > 0$ and $\{0\}/\{1\} = 0$. This is in accord with (2.10) since $\big(\tilde{\Delta}_{\pm}\big)^* = \tilde{\Delta}_{\mp} \to \varepsilon^{1/2}M_{\mp}$.

4. Tensor products of harmonic series irreducible representations of $Sp(2n,\Re)$ and their associates

As in the previous section it is straightforward to exploit Definition 3.1 and the $Sp(2n,\Re)$ tensor product rule (2.10) to establish:

Proposition 4.1

$$\langle\tfrac{1}{2}k(\mu)\rangle^* \times \langle\tfrac{1}{2}\ell(\nu)\rangle^* = \Big(\langle\tfrac{1}{2}k(\mu)\rangle \times \langle\tfrac{1}{2}\ell(\nu)\rangle\Big)^* \tag{4.1}$$

where on the left-hand side the symbols $*$ indicate k- and ℓ-associates, and on the right-hand side $(k+\ell)$-associates.

Proof It should first be noted that under the restriction from $O(k+\ell)$ to $O(k)\times O(\ell)$ we have in the notation of (2.11) and (3.1)

$$[\lambda]^* = [\lambda^*] \to \sum_{\mu,\nu} K^{\mu\nu}_{\lambda^*}\ [\mu]\times[\nu]. \tag{4.2}$$

However, (2.11) and (3.1) also imply:

$$\begin{aligned}
[\lambda]^* = [\lambda]\cdot[1^{k+\ell}] &\to \sum_{\mu,\nu} K^{\mu\nu}_{\lambda}\ ([\mu]\times[\nu])\cdot([1^k]\times[1^\ell]) \\
&= \sum_{\mu,\nu} K^{\mu\nu}_{\lambda}\ ([\mu]\cdot[1^k])\times([\nu]\cdot[1^\ell]) \\
&= \sum_{\mu,\nu} K^{\mu\nu}_{\lambda}\ [\mu]^*\times[\nu]^* = \sum_{\mu,\nu} K^{\mu^*\nu^*}_{\lambda}\ [\mu]\times[\nu],
\end{aligned} \tag{4.3}$$

where in the first step advantage has been taken of the fact that $[1^{k+\ell}](A) = \det A = \det B \det C = [1^k](B)\ [1^\ell](C)$ for any $A = B\times C$ in $O(k)\times O(\ell)$.

Comparing (4.2) and (4.3), we have

$$K^{\mu^*\nu^*}_{\lambda} = K^{\mu\nu}_{\lambda^*}. \tag{4.4}$$

It then follows that

$$\begin{aligned}
&\langle\tfrac{1}{2}k(\mu)\rangle^* \times \langle\tfrac{1}{2}\ell(\nu)\rangle^* = \langle\tfrac{1}{2}k(\mu^*)\rangle \times \langle\tfrac{1}{2}\ell(\nu^*)\rangle = \sum_{\lambda} K^{\mu^*\nu^*}_{\lambda}\ \langle\tfrac{1}{2}(k+\ell)(\lambda)\rangle \\
&= \sum_{\lambda} K^{\mu\nu}_{\lambda^*}\ \langle\tfrac{1}{2}(k+\ell)(\lambda)\rangle = \sum_{\lambda} K^{\mu\nu}_{\lambda}\ \langle\tfrac{1}{2}(k+\ell)(\lambda^*)\rangle = \sum_{\lambda} K^{\mu\nu}_{\lambda}\ \langle\tfrac{1}{2}(k+\ell)(\lambda)\rangle^* \\
&= \Big(\langle\tfrac{1}{2}k(\mu)\rangle \times \langle\tfrac{1}{2}\ell(\nu)\rangle\Big)^*,
\end{aligned} \tag{4.5}$$

as required.

5. Symmetrised powers of the basic harmonic irreducible representations of $Sp(2n,\Re)$

First of all it should be pointed out that for the harmonic or metaplectic representation $\tilde\Delta$ of $Sp(2n,\Re)$ we have:

Proposition 5.1 *The k-fold symmetrised powers of $\tilde\Delta$ are such that*

$$\big(\tilde\Delta\otimes\sigma\big)^* = \tilde\Delta\otimes\sigma' \tag{5.1}$$

for each partition σ of k.

Proof In the notation of (2.15), the branching rule for the restriction from $O(k)$ to S_k is such that

$$[\lambda]\to\sum_\tau b^\lambda_\tau\ (\tau)\quad\text{and}\quad [\lambda^*]\to\sum_\sigma b^{\lambda^*}_\sigma\ (\sigma). \tag{5.2}$$

However

$$[\lambda^*]=[\lambda]^*=[\lambda]\cdot[0^*]=[\lambda]\cdot[1^k]$$

$$\to\sum_\tau b^\lambda_\tau\ (\tau\cdot(1^k))=\sum_\tau b^\lambda_\tau\ (\tau')=\sum_\sigma b^\lambda_{\sigma'}\ (\sigma). \tag{5.3}$$

Comparing (5.2) and (5.3) gives

$$b^{\lambda^*}_\sigma=b^\lambda_{\sigma'}. \tag{5.4}$$

Using this and (2.14) we then have

$$\begin{aligned}\tilde\Delta\otimes\sigma' &= \sum_\lambda\ b^\lambda_{\sigma'}\ \langle\tfrac12 k(\lambda)\rangle=\sum_\lambda\ b^{\lambda^*}_\sigma\ \langle\tfrac12 k(\lambda)\rangle\\ &=\sum_\lambda\ b^\lambda_\sigma\ \langle\tfrac12 k(\lambda^*)\rangle=\sum_\lambda\ b^\lambda_\sigma\ \langle\tfrac12 k(\lambda)\rangle^*=\big(\tilde\Delta\otimes\sigma\big)^*,\end{aligned} \tag{5.5}$$

as required.

This result (5.1) for the metaplectic representation $\tilde\Delta$ may be refined so as to provide information on the symmetrised powers of the basic harmonic irreducible representations $\tilde\Delta_\pm$. It has been conjectured[14,15] on the basis of extensive calculations of such symmetrised powers[16] that:

Proposition 5.2 *The symmetrised k-fold powers of the basic harmonic irreducible representations $\tilde\Delta_\pm$ of $Sp(2n,\Re)$ are such that*

$$\big(\tilde\Delta_\pm\otimes\{\rho\}\big)^*=\tilde\Delta_\mp\otimes\{\rho'\}, \tag{5.6}$$

for each partition ρ of k.

In order to prove this result it is helpful first to establish two Lemmas. First of all we need a generalisation of Littlewood's conjugacy formula[26] which states that for any partition σ of k we have

$$\big(\{\sigma\}\otimes\{\rho\}\big)'=\begin{cases}\{\sigma'\}\otimes\{\rho\} & \text{if } k \text{ is even;}\\ \{\sigma'\}\otimes\{\rho'\} & \text{if } k \text{ is odd.}\end{cases} \tag{5.7}$$

The requisite generalisation of (5.7) takes the form:

Lemma 5.3 *Let S be an arbitrary representation of $U(n)$ of the form*

$$S = \sum_{\sigma} \{\sigma\} \tag{5.8}$$

where repetitions are allowed but each summand $\{\sigma\}$ has the same fixed parity η_S in the sense that if σ is a partition of k then $k \equiv \eta_S (\mathrm{mod} 2)$ with η_S fixed to be either 0 or 1. Then

$$\big(S \otimes \{\rho\}\big)' = \begin{cases} S' \otimes \{\rho\} & \text{if } \eta_S = 0; \\ S' \otimes \{\rho'\} & \text{if } \eta_S = 1, \end{cases} \tag{5.9}$$

where S' is obtained from S by conjugating each summand.

Proof The result is valid by virtue of Littlewood's conjugacy formula (5.7) if S has one summand $\{\sigma\}$. We assume that it is valid for all T with one fewer summand, say $\{\sigma\}$, than S. Writing $S = T + \{\sigma\}$ we then have

$$\begin{aligned}
\big(S \otimes \{\rho\}\big)' &= \big((T + \{\sigma\}) \otimes \{\rho\}\big)' \\
&= \big(\sum_{\mu\nu} c^{\rho}_{\mu\nu} (T \otimes \{\mu\})(\{\sigma\} \otimes \{\nu\})\big)' \\
&= \sum_{\mu\nu} c^{\rho}_{\mu\nu} (T \otimes \{\mu\})'(\{\sigma\} \otimes \{\nu\})' \\
&= \begin{cases} \sum_{\mu\nu} c^{\rho}_{\mu\nu} (T' \otimes \{\mu\})(\{\sigma'\} \otimes \{\nu\}) & \text{if } \eta_T = 0; \\ \sum_{\mu\nu} c^{\rho}_{\mu\nu} (T' \otimes \{\mu'\})(\{\sigma'\} \otimes \{\nu'\}) & \text{if } \eta_T = 1, \end{cases} \\
&= \begin{cases} \sum_{\mu\nu} c^{\rho}_{\mu\nu} (T' \otimes \{\mu\})(\{\sigma'\} \otimes \{\nu\}) & \text{if } \eta_T = 0; \\ \sum_{\mu'\nu'} c^{\rho'}_{\mu'\nu'} (T' \otimes \{\mu'\})(\{\sigma'\} \otimes \{\nu'\}) & \text{if } \eta_T = 1; \end{cases} \\
&= \begin{cases} (T' + \{\sigma'\}) \otimes \{\rho\}) & \text{if } \eta_T = 0; \\ (T' + \{\sigma'\}) \otimes \{\rho'\}) & \text{if } \eta_T = 1, \end{cases} \\
&= \begin{cases} S' \otimes \{\rho\}) & \text{if } \eta_S = 0; \\ S' \otimes \{\rho'\}) & \text{if } \eta_S = 1, \end{cases}
\end{aligned} \tag{5.10}$$

where use has been made of the fact that $\eta_S = \eta_T$. The coefficients $c^{\rho}_{\mu\nu}$ are just the Littlewood-Richardson coefficients[8,9] determined by the tensor product rule for $U(n)$:

$$\{\mu\} \cdot \{\nu\} = \sum_{\rho} c^{\rho}_{\mu\nu} \{\rho\} \tag{5.11}$$

which satisfy the conjugacy relation

$$c^{\rho'}_{\mu'\nu'} = c^{\rho}_{\mu\nu} \tag{5.12}.$$

This completes the inductive proof of Lemma 5.3.

Our second Lemma takes the form:
Lemma 5.4 *For each partition ρ of k*

$$\big(M_\pm \otimes \{\rho\}\big)/\{1^k\} = M_\mp \otimes \{\rho'\}. \tag{5.13}$$

Proof The branching rule for the restriction from $U(n)$ to $U(1) \times U(n-1)$ takes the form:

$$\{\mu\} \to \sum_{a=0}^{\mu_1} z^a \{\mu\}/\{a\} \tag{5.14}$$

where it has been convenient to denote the character $\{1\}$ of $U(1)$ simply by z, and $\{a\}$ by z^a. In the special case $\{\mu\} = \{1^m\}$ this gives

$$\{1^m\} \to \sum_{a=0}^{1} z^a \{1^m\}/\{a\} = \{1^m\} + z\{1^{m-1}\}. \tag{5.15}$$

Taking the k-fold symmetrised power specified by a partition ρ of k gives

$$\begin{aligned}\{1^m\} \otimes \{\rho\} &\to \sum_{b=0}^{k} z^b \big(\{1^m\} \otimes \{\rho\}\big)/\{b\} \\ &= \big(\{1^m\} + z\{1^{m-1}\}\big) \otimes \{\rho\}.\end{aligned} \tag{5.16}$$

Equating the coefficients of the terms in z^k gives

$$\big(\{1^m\} \otimes \{\rho\}\big)/\{k\} = \{1^{m-1}\} \otimes \{\rho\}. \tag{5.17}$$

Applying Littlewood's conjugacy formula (5.7) to both sides of (5.17) gives

$$\big(\{m\} \otimes \{\sigma\}\big)/\{1^k\} = \{m-1\} \otimes \{\sigma'\}. \tag{5.18}$$

All this can be generalised. If we set $Q_\pm = M'_\pm$ so that

$$Q_+ = \sum_{m:m \text{ even}} \{1^m\} = \{0\} + \{1^2\} + \{1^4\} + \cdots; \tag{5.19a}$$

$$Q_- = \sum_{m:m \text{ odd}} \{1^m\} = \{1\} + \{1^3\} + \{1^5\} + \cdots, \tag{5.19b}$$

then under the restriction $U(n) \to U(1) \times U(n-1)$

$$Q_\pm \to \sum_{a=0}^{1} z^a Q_\pm/\{a\} = Q_\pm + zQ_\mp \tag{5.20}$$

and hence

$$Q_\pm \otimes \{\rho\} \to \sum_{b=0}^{k} z^b \big(Q_\pm \otimes \{\rho\}\big)/\{b\} = \big(Q_\pm + zQ_\mp\big) \otimes \{\rho\}. \tag{5.21}$$

Once again equating the coefficients of the terms in z^k gives

$$(Q_\pm \otimes \{\rho\})/\{k\} = \{Q_\mp \otimes \{\rho\}. \tag{5.22}$$

Our required result (5.13) then follows from our conjugacy Lemma 5.3 since the terms of Q_+ are of parity $\eta_{Q_+} = 0$ and those of Q_- are of parity $\eta_{Q_-} = 1$, while $Q'_\pm = M_\pm$.

Armed with Lemma 5.4 we are now in a position to prove Proposition 5.2:

Proof For any partition ρ of k all the irreducible representations in the k-fold symmetrised power $\tilde{\Delta}_\pm \otimes \{\rho\}$ are of the form $\langle \frac{1}{2}k(\lambda)\rangle$. It then follows from (2.10), Proposition 3.2 and Lemma 5.4 that

$$\begin{aligned}\left(\tilde{\Delta}_\pm \otimes \{\rho\}\right)^* &\to \left((\varepsilon^{1/2}M_\pm) \otimes \{\rho\}\right)/\{1^k\} \\ &= \varepsilon^{k/2}\left(M_\pm \otimes \{\rho\}\right)/\{1^k\} = \varepsilon^{k/2}\left(M_\mp \otimes \{\rho'\}\right)\end{aligned} \tag{5.23}$$

Comparing this with

$$\tilde{\Delta}_\mp \otimes \{\rho'\} \to (\varepsilon^{1/2}M_\mp) \otimes \{\rho'\} = \varepsilon^{k/2}\left(M_\mp \otimes \{\rho'\}\right) \tag{5.24}$$

suffices to prove (5.6).

Remarkably, as indicated through the calculation of numerous examples[16]. Proposition 5.2, may be generalised to give:

Proposition 5.5 *For any partition ρ of r, the corresponding r-fold symmetrised power of the associate irreducible representation $\langle \frac{1}{2}k(\lambda)\rangle^*$ of $Sp(2n, \Re)$ is such that*

$$\langle \tfrac{1}{2}k(\lambda)\rangle^* \otimes \{\rho\} = \begin{cases} \left(\langle \frac{1}{2}k(\lambda)\rangle \otimes \{\rho\}\right)^* & \text{if } k \text{ is even;} \\ \left(\langle \frac{1}{2}k(\lambda)\rangle \otimes \{\rho'\}\right)^* & \text{if } k \text{ is odd,} \end{cases} \tag{5.25}$$

where the * on the left signifies a k-associate, while those on the right signify kr-associates.

To prove this Proposition the first task is to generalise Lemma 5.4:

Lemma 5.6 *Let S be an arbitrary representation of $U(n)$ of the form*

$$S = \sum_{\sigma:\sigma'_1 \le k} \{\sigma\} \tag{5.26}$$

where repetitions are allowed but each summand $\{\sigma\}$ has the same fixed parity η_S and $\sigma'_1 \le k$. Then for each partition ρ of r:

$$(S/\{1^k\}) \otimes \{\rho\} = \begin{cases} (S \otimes \{\rho\})/\{1^{kr}\} & \text{if } k \text{ is even;} \\ (S \otimes \{\rho'\})/\{1^{kr}\} & \text{if } k \text{ is odd.} \end{cases} \tag{5.27}$$

Proof Let $\{\mu\}$ be an irreducible representation of $U(n)$ with $\mu_1 \le k$ and μ a partition of m. Then taking the r-fold symmetrised power of $\{\mu\}$ specified by ρ and restricting from $U(n)$ to $U(1) \times U(n-1)$ as in (5.14) gives

$$\sum_{b=0}^{kr} z^b \left(\{\mu\} \otimes \{\rho\}\right)/\{b\} = \left(\sum_{a=0}^{k} z^a \{\mu\}/\{a\}\right) \otimes \{\rho\}. \tag{5.27}$$

Comparing terms in z^{kr} on both sides of this equation gives the identity

$$(\{\mu\}/\{k\}) \otimes \{\rho\} = (\{\mu\} \otimes \{\rho\})/\{kr\}. \tag{5.28}$$

Taking the conjugate of the left hand side gives:

$$\big((\{\mu\}/\{k\}) \otimes \{\rho\}\big)' = \begin{cases} (\{\mu'\}/\{1^k\}) \otimes \{\rho\} & \text{if } m-k \text{ is even;} \\ (\{\mu'\}/\{1^k\}) \otimes \{\rho'\} & \text{if } m-k \text{ is odd,} \end{cases} \tag{5.29}$$

while the conjugate of the right hand side gives:

$$\big((\{\mu\} \otimes \{\rho\})/\{kr\}\big)' = \begin{cases} (\{\mu'\} \otimes \{\rho\})/\{1^{kr}\} & \text{if } m \text{ is even;} \\ (\{\mu'\} \otimes \{\rho'\})/\{1^{kr}\} & \text{if } m \text{ is odd.} \end{cases} \tag{5.30}$$

Comparing (5.29) and (5.30) and setting $\sigma = \mu'$ gives the conjugate of (5.28), namely

$$(\{\sigma\}/\{1^k\}) \otimes \{\rho\} = \begin{cases} (\{\sigma\} \otimes \{\rho\})/\{1^{kr}\} & \text{if } k \text{ is even;} \\ (\{\sigma\} \otimes \{\rho'\})/\{1^{kr}\} & \text{if } k \text{ is odd.} \end{cases} \tag{5.31}$$

It should be recalled that this only applies if $\sigma_1' = \mu_1 \leq k$. However, by hypothesis all the summands $\{\sigma\}$ of S in (5.26) are of this type. Moreover all the summands are of the same parity η_S. This allows us to replace $\{\mu\} = \{\sigma'\}$ by S' in both (5.27) and (5.28) to give

$$\sum_{b=0}^{kr} z^b (S' \otimes \{\rho\})/\{b\} = \left(\sum_{a=0}^{k} z^a S'/\{a\}\right) \otimes \{\rho\}. \tag{5.32}$$

and

$$(S'/\{k\}) \otimes \{\rho\} = (S' \otimes \{\rho\})/\{kr\}. \tag{5.33}$$

Setting $T = S'/\{k\}$ so that $\eta_T = \eta_S$ if k is even and $\eta_T = 1 - \eta_S$ if k is odd, it then follows from Lemma 5.3 that taking the conjugate of the left hand side of (5.33) gives:

$$\big((S'/\{k\}) \otimes \{\rho\}\big)' = \begin{cases} (S/\{1^k\}) \otimes \{\rho\} & \text{if } \eta_T = 0; \\ (S/\{1^k\}) \otimes \{\rho'\} & \text{if } \eta_T = 1. \end{cases} \tag{5.34}$$

Similarly from Lemma 5.3 taking the conjugate of the right hand side of (5.33) gives:

$$\big((S' \otimes \{\rho\})/\{kr\}\big)' = \begin{cases} (S \otimes \{\rho\})/\{1^{kr}\} & \text{if } \eta_S = 0; \\ (S \otimes \{\rho'\})/\{1^{kr}\} & \text{if } \eta_S = 1. \end{cases} \tag{5.35}$$

Comparing (5.34) and (5.35) gives the conjugate of (5.33), namely

$$(S/\{1^k\}) \otimes \{\rho\} = \begin{cases} (S \otimes \{\rho\})/\{1^{kr}\} & \text{if } k \text{ is even;} \\ (S \otimes \{\rho'\})/\{1^{kr}\} & \text{if } k \text{ is odd,} \end{cases} \tag{5.36}$$

as required in order to prove Lemma 5.6

This now allows us to prove Proposition 5.5

Proof First of all, under the restriction $Sp(2n,\Re)\to U(n)$ we have from (2.4)

$$\langle\tfrac{1}{2}k(\lambda)\rangle \to \sum_\mu \varepsilon^{k/2}R^\mu_\lambda\{\mu\} = \varepsilon^{k/2}S \tag{5.37}$$

with S as in Lemma 5.6. It follows that

$$\langle\tfrac{1}{2}k(\lambda)\rangle\otimes\{\rho\} \to \left(\varepsilon^{k/2}S\right)\otimes\{\rho\} = \varepsilon^{kr/2}S\otimes\{\rho\} \tag{5.38}$$

Taking the k-associate of (5.37) and using Proposition 3.2 then gives under the same restriction from $Sp(2n,\Re)\to U(n)$:

$$\langle\tfrac{1}{2}k(\lambda)\rangle^* \to \sum_\mu \varepsilon^{k/2}R^\mu_\lambda\{\mu\}/\{1^k\} = \varepsilon^{k/2}S/\{1^k\}. \tag{5.39}$$

Taking the r-fold symmetrised product of (5.39) specified by the partition ρ and using Lemma 5.6 then gives

$$\begin{aligned}\langle\tfrac{1}{2}k(\lambda)\rangle^*\otimes\{\rho\} \to& \varepsilon^{kr/2}\left(S/\{1^k\}\right)\otimes\{\rho\}\\ &= \begin{cases}\varepsilon^{kr/2}(S\otimes\{\rho\})/\{1^{kr}\} & \text{if } k \text{ is even;}\\ \varepsilon^{kr/2}(S\otimes\{\rho'\})/\{1^{kr}\} & \text{if } k \text{ is odd.}\end{cases}\end{aligned} \tag{5.40}$$

On the other hand taking the kr-associate of (5.38) and using Proposition 3.2 gives

$$\left(\langle\tfrac{1}{2}k(\lambda)\rangle\otimes\{\rho\}\right)^* \to \varepsilon^{kr/2}(S\otimes\{\rho\})/\{1^{kr}\}. \tag{5.41}$$

Replacing ρ by ρ' then gives

$$\left(\langle\tfrac{1}{2}k(\lambda)\rangle\otimes\{\rho'\}\right)^* \to \varepsilon^{kr/2}(S\otimes\{\rho'\})/\{1^{kr}\} \tag{5.42}$$

Hence comparing (5.40) with (5.41) and (5.42) it follows that

$$\langle\tfrac{1}{2}k(\lambda)\rangle^*\otimes\{\rho\} = \begin{cases}\left(\langle\tfrac{1}{2}k(\lambda)\rangle\otimes\{\rho\}\right)^* & \text{if } k \text{ is even;}\\ \left(\langle\tfrac{1}{2}k(\lambda)\rangle\otimes\{\rho'\}\right)^* & \text{if } k \text{ is odd,}\end{cases} \tag{5.43}$$

as required.

6. Symmetrised powers of arbitrary harmonic series irreducible representations of $Sp(2n,\Re)$

It is possible to exploit the remarks following (2.6) and the branching rule (2.9) from $Sp(2n,\Re)$ to $U(n)$ to derive the following general formula for symmetrised powers or plethysms of arbitrary harmonic series irreducible representations of $Sp(2n,\Re)$:

Proposition 6.1 *Let the partition λ be such that $\lambda'_1+\lambda'_2\le k$ and $\lambda'_1\le n$ and let ρ be an arbitrary partition of r, then*

$$\langle\tfrac{1}{2}k(\lambda)\rangle\otimes\{\rho\} = \sum_\mu x^\mu_{\lambda\rho}\langle\tfrac{1}{2}kr(\mu)\rangle, \tag{6.1}$$

where the summation is over all partitions μ satisfying the constraints $\mu_1'+\mu_2'\le kr$ and $\mu_1'\le n$, and the coefficients $x^{\mu}_{\lambda\rho}$ are determined by the expansion

$$((\{\lambda_s\}^k\cdot D)\otimes\{\rho\})\cdot C=\sum_{\mu} x^{\mu}_{\lambda\rho}\{\mu_s\}^{kr} \tag{6.2}$$

with $C=D^{-1}$.

Proof Under the restriction from $Sp(2n,\Re)$ to $U(n)$ the branching rule (2.9) takes the form:

$$\langle\tfrac{1}{2}k(\lambda)\rangle\to\varepsilon^{k/2}\cdot\{\lambda_s\}^k\cdot D. \tag{6.3}$$

Hence, for each partition ρ of r, the corresponding r-fold symmetrised power of this irreducible representation decomposes in accordance with the formula

$$\begin{aligned}\langle\tfrac{1}{2}k(\lambda)\rangle\otimes\{\rho\} &\to \left(\varepsilon^{k/2}\cdot\{\lambda_s\}^k\cdot D\right)\otimes\{\rho\}\\ &=\varepsilon^{kr/2}\cdot((\{\lambda_s\}^k\cdot D)\otimes\{\rho\})\\ &=\varepsilon^{kr/2}\cdot((\{\lambda_s\}^k\cdot D)\otimes\{\rho\})\cdot D^{-1}D\\ &=\varepsilon^{kr/2}\cdot(((\{\lambda_s\}^k\cdot D)\otimes\{\rho\})\cdot D^{-1})\cdot D.\end{aligned} \tag{6.4}$$

However, in the notation of (6.2), it follows once again from the branching rule (2.9) that

$$\langle\tfrac{1}{2}k(\lambda)\rangle\otimes\{\rho\}=\sum_{\mu}x^{\mu}_{\lambda\rho}\langle\tfrac{1}{2}kr(\mu)\rangle\to\varepsilon^{kr/2}\left(\sum_{\mu}x^{\mu}_{\lambda\rho}\{\mu_s\}^{kr}\right)\cdot D. \tag{6.5}$$

Comparison of (6.4) and (6.5) then completes the proof since, as noted following (2.6), identities at the level of their $U(n)$ content is sufficient to imply identities between representations of $Sp(2n,\Re)$.

In making use of the formula (6.2) to evaluate the plethysm coefficients in (6.1) it is possible to make one or two simplifications. While the product of the signed sequence and D-series appearing in the branching rule (2.9) is a product of two infinite series, all surviving terms $\{\nu\}$ in the product will *automatically* be such that $\nu_1'\le k$. Since the products are carried out in $U(n)$ all the surviving terms are also *automatically* such that $\nu_1'\le n$. It follows that (2.9) is equivalent to[3]

$$\langle\tfrac{1}{2}k(\lambda)\rangle\to\varepsilon^{k/2}\cdot\left(\{\lambda_s\}^k_N\cdot D_N\right)_N \tag{6.6}$$

where $N=\min(n,k)$, with the various series and products all being evaluated in $U(N)$. In precisely the same way the plethysm and subsequent product with C in (6.2) may be evaluated in $U(M)$ where $M=\min(kr,n)$ so that (6.2) may be replaced by

$$\left(((\{\lambda_s\}^k_N\cdot D_N)_N\otimes\{\rho\})_M\cdot C_M\right)_M=\sum_{\mu}x^{\mu}_{\lambda\rho}\{\mu_s\}^{kr}_M. \tag{6.7}$$

Finally, it should be noted that in order to read off the required plethysm coefficients from the expansion (6.7) it is only necessary to look at the leading term $\{\mu\}$ in each signed sequence $\{\mu_s\}^{kr}_M$, since it is only the leading term of each signed sequence which satisfies the required $O(kr)$-standardness condition $\mu'_1 + \mu'_2 \leq kr$.

We illustrate the diverse features of such calculations by the evaluation of the plethysm $\langle 2(21)\rangle \otimes \{21\}$ for $Sp(24,\Re)$ as an explicit expansion in terms of irreducible representations of the form $\langle 6(\mu)\rangle$ with the partition μ restricted, for convenience, to have weight ≤ 18 and width ≤ 3. Here we have $k = 4$ and $n = 12$ so that $N = \min(k,n) = 4$. Hence the signed sequence, evaluated using the modification rules of $O(4)$, but restricted to terms standard in $U(4)$, has just the two terms

$$\{21_s\}^4_4 = \{21\} - \{2^3 1\}, \tag{6.8}$$

both of which have width ≤ 3. The terms in the D-series restricted to width ≤ 3 and length ≤ 4 are

$$\{0\} + \{2\} + \{2^2\} + \{2^3\} + \{2^4\}. \tag{6.9}$$

Evaluation, in $U(4)$, of the tensor product of (6.8) with (6.9) yields the terms of width ≤ 3 as

$$A = \{21\} + \{2^2 1\} + \{31^2\} + \{32\} + \{321^2\} + \{32^2\} + \{3^2 1\} + \{3^2 21\}. \tag{6.10}$$

The plethysm of $A \otimes \{21\}$ is now to be evaluated in the group $U(12)$ since $k = 4$, $r = 3$ and $n = 12$ so that $M = \min(kr,n) = 12$. Keeping all terms of width ≤ 3 and of weight ≤ 18 gives:

$$\begin{array}{lllll}
\{2^4 1\} & + \{2^4 1^3\} & + 2\{2^5 1\} & + \{2^5 1^3\} & + 2\{2^6 1\} \\
+ \{2^7 1\} & + \{321^4\} & + 2\{32^2 1^2\} & + 3\{32^2 1^4\} & + \{32^3\} \\
+ 9\{32^3 1^2\} & + 5\{32^3 1^4\} & + 6\{32^4\} & + 15\{32^4 1^2\} & + 4\{32^4 1^4\} \\
+ 10\{32^5\} & + 11\{32^5 1^2\} & + \{32^5 1^4\} & + 7\{32^6\} & + 3\{32^6 1^2\} \\
+ 2\{32^7\} & + \{3^2 1^3\} & + 2\{3^2 1^5\} & + 3\{3^2 21\} & + 12\{3^2 21^3\} \\
+ 7\{3^2 21^5\} & + 18\{3^2 2^2 1\} & + 33\{3^2 2^2 1^3\} & + 9\{3^2 2^2 1^5\} & + 45\{3^2 2^3 1\} \\
+ 40\{3^2 2^3 1^3\} & + 5\{3^2 2^3 1^5\} & + 54\{3^2 2^4 1\} & + 23\{3^2 2^4 1^3\} & + 31\{3^2 2^5 1\} \\
+ 12\{3^3 1^2\} & + 20\{3^3 1^4\} & + 5\{3^3 1^6\} & + 10\{3^3 2\} & + 60\{3^3 21^2\} \\
+ 51\{3^3 21^4\} & + 7\{3^3 21^6\} & + 40\{3^3 2^2\} & + 117\{3^3 2^2 1^2\} & + 51\{3^3 2^2 1^4\} \\
+ 71\{3^3 2^3\} & + 111\{3^3 2^3 1^2\} & + 67\{3^3 2^4\} & + 32\{3^4 1\} & + 70\{3^4 1^3\} \\
+ 31\{3^4 1^5\} & + 120\{3^4 21\} & + 137\{3^4 21^3\} & + 181\{3^4 2^2 1\} & + 28\{3^5\} \\
+ 116\{3^5 1^2\} & + 92\{3^5 2\} & & &
\end{array} \tag{6.11}$$

We now form the tensor product, in $U(12)$, of the above terms with the following terms of width ≤ 3 of the C-series:

$$\{0\} - \{2\} + \{31\} - \{3^2\}. \tag{6.12}$$

Keeping only terms in the tensor product up to width 3 and weight 18 yields

$$\begin{array}{lllll}
\{2^4 1\} & + \{2^4 1^3\} & + \{2^5 1\} & - \{2^6 1^3\} & - \{2^7 1\} \\
- \{2^8 1\} & + \{321^4\} & + 2\{32^2 1^2\} & + 2\{32^2 1^4\} & + \{32^3\} \\
+ 6\{32^3 1^2\} & + \{32^3 1^4\} & + 4\{32^4\} & + 4\{32^4 1^2\} & - \{32^4 1^4\} \\
+ 3\{32^5\} & - 4\{32^5 1^2\} & - 2\{32^5 1^4\} & - 3\{32^6\} & - 6\{32^6 1^2\} \\
- 4\{32^7\} & + \{3^2 1^3\} & + \{3^2 1^5\} & + 3\{3^2 21\} & + 8\{3^2 21^3\} \\
+ 3\{3^2 21^5\} & + 12\{3^2 2^2 1\} & + 12\{3^2 2^2 1^3\} & + 16\{3^2 2^3 1\} & - 3\{3^2 2^3 1^5\} \\
- 12\{3^2 2^4 1^3\} & - 16\{3^2 2^5 1\} & + 8\{3^3 1^2\} & + 8\{3^3 1^4\} & + 7\{3^3 2\} \\
+ 24\{3^3 21^2\} & + 7\{3^3 21^4\} & + 16\{3^3 2^2\} & + 16\{3^3 2^2 1^2\} & - 7\{3^3 2^2 1^4\} \\
+ 9\{3^3 2^3\} & - 16\{3^3 2^3 1^2\} & - 9\{3^3 2^4\} & + 13\{3^4 1\} & + 13\{3^4 1^3\} \\
+ 25\{3^4 21\} & + 6\{3^5\} & + 6\{3^5 1^2\} & + 6\{3^5 2\} &
\end{array} \tag{6.13}$$

The terms may now be grouped together into sets of $O(12)$ signed sequences. Thus, for example, $\{2^41_s\}_{12}^{12} = \{2^41\} - \{2^81\}$. Alternatively, bearing in mind that for the purposes of determining plethysm coefficients it is only necessary to retain the leading $O(12)$-standard term in each such signed sequence, (6.13) may simply be restricted to those terms $\{\mu\}$ for which $\mu_1' + \mu_2' \leq 12$. The surviving terms are

$$
\begin{array}{lllll}
\{2^41\} & +\{2^41^3\} & +\{2^51\} & +\{321^4\} & +2\{32^21^2\} \\
+2\{32^21^4\} & +\{32^3\} & +6\{32^31^2\} & +\{32^31^4\} & +4\{32^4\} \\
+4\{32^41^2\} & +3\{32^5\} & +\{3^21^3\} & +\{3^21^5\} & +3\{3^221\} \\
+8\{3^221^3\} & +3\{3^221^5\} & +12\{3^22^21\} & +12\{3^22^21^3\} & +16\{3^22^31\} \\
+8\{3^31^2\} & +8\{3^31^4\} & +7\{3^32\} & +24\{3^321^2\} & +7\{3^321^4\} \\
+16\{3^32^2\} & +16\{3^32^21^2\} & +9\{3^32^3\} & +13\{3^41\} & +13\{3^41^3\} \\
+25\{3^421\} & +6\{3^5\} & +6\{3^51^2\} & +6\{3^52\} &
\end{array}
\qquad (6.14)
$$

These irreducible representations of $U(12)$ can now be converted back into the irreducible representations of $Sp(24,\Re)$, to which they correspond in a one-to-one manner, by the simple insertion of a 6 and a change to $Sp(24,\Re)$ notation to give

$$
\begin{array}{llll}
<6(2^41)> & +<6(2^41^3)> & +<6(2^51)> & +<6(321^4)> \\
+2<6(32^21^2)> & +2<6(32^21^4)> & +<6(32^3)> & +6<6(32^31^2)> \\
+<6(32^31^4)> & +4<6(32^4)> & +4<6(32^41^2)> & +3<6(32^5)> \\
+<6(3^21^3)> & +<6(3^21^5)> & +3<6(3^221)> & +8<6(3^221^3)> \\
+3<6(3^221^5)> & +12<6(3^22^21)> & +12<6(3^22^21^3)> & +16<6(3^22^31)> \\
+8<6(3^31^2)> & +8<6(3^31^4)> & +7<6(3^32)> & +24<6(3^321^2)> \\
+7<6(3^321^4)> & +16<6(3^32^2)> & +16<6(3^32^21^2)> & +9<6(3^32^3)> \\
+13<6(3^41)> & +13<6(3^41^3)> & +25<6(3^421)> & +6<6(3^5)> \\
+6<6(3^51^2)> & +6<6(3^52)> & &
\end{array}
\qquad (6.15)
$$

It follows that up to weight 18 and width 3 the required plethysm takes the form:

$$
\begin{array}{llll}
\langle 2(21)\rangle \otimes \{21\} & & & \\
=<6(2^41)> & +<6(2^41)^*> & +<6(2^51)> & +<6(321^4)> \\
+2<6(32^21^2)> & +2<6(32^21^2)^*> & +<6(32^3)> & +<6(32^3)^*> \\
+6<6(32^31^2)> & +4<6(32^4)> & +4<6(32^4)^*> & +3<6(32^5)> \\
+<6(3^21^3)> & +<6(3^21^3)^*> & +3<6(3^221)> & +3<6(3^221)^*> \\
+8<6(3^221^3)> & +12<6(3^22^21)> & +12<6(3^22^21)^*> & +16<6(3^22^31)> \\
+8<6(3^31^2)> & +8<6(3^31^2)^*> & +7<6(3^32)> & +7<6(3^32)^*> \\
+24<6(3^321^2)> & +16<6(3^32^2)> & +16<6(3^32^2)^*> & +9<6(3^32^3)> \\
+13<6(3^41)> & +13<6(3^41)^*> & +25<6(3^421)> & +6<6(3^5)> \\
+6<6(3^5)^*> & +6<6(3^52)> & +\cdots &
\end{array}
\qquad (6.16)
$$

where the terms have now been arranged in mutually associated pairs of irreducible representations together with self-associate irreducible representations, so as to illustrate in accordance with Proposition 5.5 the self-associate nature of this particular plethysm.

7. The analogy between $Sp(2n,\Re)$ and $SO(2n)$

In $SO(2n)$ there exists the basic spin representation $\Delta = \Delta_+ + \Delta_-$ which is a direct sum of the two irreducible representations Δ_+ and Δ_- whose branchings from

$SO(2n)$ to $U(n)$ take the form

$$\Delta_+ \to \varepsilon^{-1/2} \sum_{x=0} \{1^{n-2x}\}; \tag{7.1a}$$

$$\Delta_- \to \varepsilon^{-1/2} \sum_{x=0} \{1^{n-1-2x}\}. \tag{7.1b}$$

As we have seen for $Sp(2n,\Re)$ there exists the basic harmonic representation $\tilde{\Delta} = \tilde{\Delta}_+ + \tilde{\Delta}_-$ which is a direct sum of the two irreps $\tilde{\Delta}_+$ and $\tilde{\Delta}_-$ whose branchings (2.10) from $Sp(2n,\Re)$ to $U(n)$ can be written in a form strikingly similar to (7.1):

$$\tilde{\Delta}_+ \to \varepsilon^{1/2} \sum_{x=0} \{2x\}; \tag{7.2a}$$

$$\tilde{\Delta}_- \to \varepsilon^{1/2} \sum_{x=0} \{2x+1\}. \tag{7.2b}$$

Moving to symmetrised squares, for $SO(2n)$ we have[27]:

$$\Delta_+ \otimes \{2\} \ = [1^n]_+ + \sum_{x=0} [1^{n-4-4x}]; \tag{7.3a}$$

$$\Delta_+ \otimes \{1^2\} = \sum_{x=0} [1^{n-2-4x}]; \tag{7.3b}$$

$$\Delta_- \otimes \{2\} \ = [1^n]_- + \sum_{x=0} [1^{n-4-4x}]; \tag{7.3c}$$

$$\Delta_- \otimes \{1^2\} = \sum_{x=0} [1^{n-2-4x}], \tag{7.3d}$$

while for $Sp(2n,\Re)$ the analogous symmetrised squares take the form[14]:

$$\tilde{\Delta}_+ \otimes \{2\} \ = \langle 1(0)\rangle + \sum_{x=0} \langle 1(4+4x)\rangle; \tag{7.4a}$$

$$\tilde{\Delta}_+ \otimes \{1^2\} = \sum_{x=0} \langle 1(2+4x\rangle; \tag{7.4b}$$

$$\tilde{\Delta}_- \otimes \{2\} \ = \sum_{x=0} \langle 1(2+4x\rangle; \tag{7.4c}$$

$$\tilde{\Delta}_- \otimes \{1^2\} = \langle 1(1)\rangle + \sum_{x=0} \langle 1(4+4x)\rangle. \tag{7.4d}$$

Moving to symmetrised cubes for $SO(2n)$ it is straightforward to show from previously published results[27] that we have

$$\Delta_+ \otimes \{3\} \ = \sum_{x=0} \sum_{y=0}^{11} (m_y + x) [\Delta; 1^{n-y-12x}]_{(-)^y} \text{ with } m = (100010101110); \tag{7.5a}$$

$$\Delta_+ \otimes \{21\} = \sum_{x=0} \sum_{y=0}^{5} (m_y + x) [\Delta; 1^{n-y-6x}]_{(-)^y} \text{ with } m = (0010111); \tag{7.5b}$$

$$\Delta_+ \otimes \{1^3\} = \sum_{x=0}\sum_{y=0}^{11}(m_y{+}x)[\Delta;1^{n-y-12x}]_{(-)^y} \text{ with } m=(000100110111); \tag{7.5c}$$

$$\Delta_- \otimes \{3\} = \sum_{x=0}\sum_{y=0}^{11}(m_y{+}x)[\Delta;1^{n-y-12x}]_{-(-)^y} \text{ with } m=(100010101110); \tag{7.5d}$$

$$\Delta_- \otimes \{21\} = \sum_{x=0}\sum_{y=0}^{5}(m_y{+}x)[\Delta;1^{n-y-6x}]_{-(-)^y} \text{ with } m=(0010111); \tag{7.5e}$$

$$\Delta_- \otimes \{1^3\} = \sum_{x=0}\sum_{y=0}^{11}(m_y{+}x)[\Delta;1^{n-y-12x}]_{-(-)^y} \text{ with } m=(000100110111). \tag{7.5f}$$

Encouraged by the analogy between (7.1) and (7.2), and that between (7.3) and (7.4), it seems appropriate to ask if there is a corresponding $Sp(2n,\Re)$ analogue of (7.5). The existence of such an analogue appears to be borne out by recent calculations[17].

As a warming up exercise we consider the symmetrised squares of the metaplectic representation $\tilde{\Delta}$. It follows from (2.14) with $k=2$ that:

$$\tilde{\Delta} \otimes \{\rho\} = \sum_{\lambda} b_\rho^\lambda \, \langle 1(\lambda)\rangle \tag{7.6}$$

where $\rho = (2)$ or (1^2) and λ is necessarily constrained to be either (0), $(1^2) = (0)^*$, or $(m) = (m)^*$ for $m \geq 1$, where * signifies 2-associates so that (m) is self-associate. The Coefficients b_ρ^λ are determined by the branching rule (2.16) applied to $O(2) \to S_2$:

$$[0] \to (2), \qquad [0]^* \to (1^2), \quad \text{and} \quad [m] \to (2) + (1^2), \tag{7.7}$$

where these branchings can be obtained by noting from (2.17) that:

$$[m] \to (1^2) \otimes \{m/G\} = (1^2) \otimes \big(\{m\} + \{m-1\}\big) = (1^2)^m + (1^2)^{m-1} \tag{7.8}$$

and the fact that $(1^2)^n = (0)$ for n even and $(1^2)^n = (1^2) = (0)^*$ for n odd.

It then follows from (2.14) that

$$\tilde{\Delta} \otimes \{2\} = \langle 1(0)\rangle + \sum_{m=1} \langle 1(m)\rangle; \tag{7.9a}$$

$$\tilde{\Delta} \otimes \{1^2\} = \langle 1(0)\rangle^* + \sum_{m=1} \langle 1(m).\rangle \tag{7.9b}$$

The problem of evaluating symmetrised cubes of $\tilde{\Delta}$ may be tackled in the same way. For this case $k=3$ and it is only necessary to consider only the $O(3)$ irreps $[\lambda] = [0]$, $[1^3] = [0]^*$ and $[m]$ and $[m,1] = [m]^* = [m][0]^*$ with $m = 1, 2, \ldots$ and their branching to S_3. Under the restriction $O(3) \to S_3$ we have $[0] \to (0)$ and $[0]^* = [1^3] \to (1^3)$, while the analogue of (7.8) is

$$\begin{aligned}[m] \to (21) \otimes \{m/G\} &= (21) \otimes \big(\{m\} + \{m-1\}\big) \\ &= (21) \otimes \{m\} + (21) \otimes \{m-1\}\end{aligned} \tag{7.10}$$

However

$$(21)\otimes\{n\} = \begin{cases} (1{+}x).(3) + 2x.(21) + x.(1^3) & \text{for } n{=}0{+}6x \\ x.(3) + (1{+}2x).(21) + x.(1^3) & \text{for } n{=}1{+}6x \\ (1{+}x).(3) + (1{+}2x).(21) + x.(1^3) & \text{for } n{=}2{+}6x \\ (1{+}x).(3) + (1{+}2x).(21) + (1{+}x).(1^3) & \text{for } n{=}3{+}6x \\ (1{+}x).(3) + (2{+}2x).(21) + x.(1^3) & \text{for } n{=}4{+}6x \\ (1{+}x).(3) + (2{+}2x).(21) + (1{+}x).(1^3) & \text{for } n{=}5{+}6x, \end{cases}$$

so that

$$(21)\otimes(\{m\}+\{m-1\}) = \begin{cases} (1{+}2x).(3) + 4x.(21) + 2x.(1^3) & m{=}0{+}6x \\ (1{+}2x).(3) + (1{+}4x).(21) + 2x.(1^3) & m{=}1{+}6x \\ (1{+}2x).(3) + (2{+}4x).(21) + 2x.(1^3) & m{=}2{+}6x \\ (2{+}2x).(3) + (2{+}4x).(21) + (1{+}2x).(1^3) & m{=}3{+}6x \\ (2{+}2x).(3) + (3{+}4x).(21) + (1{+}2x).(1^3) & m{=}4{+}6x \\ (2{+}2x).(3) + (4{+}4x).(21) + (1{+}2x).(1^3) & m{=}5{+}6x. \end{cases} \tag{7.11}$$

Hence:

$$[m] \to (1+[\frac{m}{3}])(3) + (m-[\frac{m}{3}])(21) + ([\frac{m}{3}])(1^3). \tag{7.12}$$

Since $[0]^* \to (1^3)$ and multiplication by (1^3) in S_3 simply involves conjugation, we have

$$[m]^* \to ([\frac{m}{3}])(3) + (m-[\frac{m}{3}])(21) + (1+[\frac{m}{3}])(1^3). \tag{7.13}$$

This completes the derivation of the $O(3)\supset S_3$ branching rules:

$$[0] \;\; \to (3); \tag{7.14a}$$
$$[0]^* \to (1^3); \tag{7.14b}$$
$$[m] \;\; \to (1+[\frac{m}{3}])(3) + (m-[\frac{m}{3}])(21) + ([\frac{m}{3}])(1^3); \tag{7.14c}$$
$$[m]^* \to ([\frac{m}{3}])(3) + (m-[\frac{m}{3}])(21) + (1+[\frac{m}{3}])(1^3). \tag{7.14d}$$

It then follows from (2.14) that

$$\tilde\Delta\otimes\{3\} \;\; = \sum_{m=0}(1+[\frac{m}{3}])\langle\tfrac{3}{2}(m)\rangle + ([\frac{m}{3}])\langle\tfrac{3}{2}(m)\rangle^*; \tag{7.15a}$$
$$\tilde\Delta\otimes\{21\} = \sum_{m=0}(m-[\frac{m}{3}])\langle\tfrac{3}{2}(m)\rangle + (m-[\frac{m}{3}])\langle\tfrac{3}{2}(m)\rangle^*; \tag{7.15b}$$
$$\tilde\Delta\otimes\{1^3\} = \sum_{m=0}([\frac{m}{3}])\langle\tfrac{3}{2}(m)\rangle + (1+[\frac{m}{3}])\langle\tfrac{3}{2}(m)\rangle^*. \tag{7.15c}$$

However

$$
\begin{aligned}
\tilde\Delta\otimes\{3\} &= \Big(\tilde\Delta_+\otimes\{3\}+\tilde\Delta_+(\tilde\Delta_-\otimes\{2\})\Big)\\
&\quad+\Big(\tilde\Delta_-\otimes\{3\}+(\tilde\Delta_+\otimes\{2\})\tilde\Delta_-\Big); && (7.16a)\\
\tilde\Delta\otimes\{21\} &= \Big(\tilde\Delta_+\otimes\{21\}+\tilde\Delta_+(\tilde\Delta_-\otimes\{2\})+\tilde\Delta_+(\tilde\Delta_-\otimes\{1^2\})\Big)\\
&\quad+\Big(\tilde\Delta_-\otimes\{21\}+(\tilde\Delta_+\otimes\{2\})\Big)\tilde\Delta_-\\
&\quad+(\tilde\Delta_+\otimes\{1^2\})\tilde\Delta_-; && (7.16b)\\
\tilde\Delta\otimes\{1^3\} &= \Big(\tilde\Delta_+\otimes\{1^3\}+\tilde\Delta_+(\tilde\Delta_-\otimes\{1^2\})\Big)\\
&\quad+\Big(\tilde\Delta_-\otimes\{1^3\}+(\tilde\Delta_+\otimes\{1^2\})\tilde\Delta_-\Big). && (7.16c)
\end{aligned}
$$

where each expression has been separated into the sum of two parts, the first of which consists of even weight terms and the second of odd weight terms. Moreover

$$
\begin{aligned}
\tilde\Delta_+(\tilde\Delta_-\otimes\{2\}) &= \sum_{i\geq 0,j\geq 0}\langle\tfrac32(2+2i+4j)\rangle+\langle\tfrac32(3+2i+4j)\rangle^*; && (7.17a)\\
\tilde\Delta_+(\tilde\Delta_-\otimes\{1^2\}) &= \sum_{i\geq 0,j\geq 0}\langle\tfrac32(4+2i+4j)\rangle+\langle\tfrac32(1+2i+4j)\rangle^*; && (7.17b)\\
\tilde\Delta_-(\tilde\Delta_-\otimes\{2\}) &= \sum_{i\geq 0,j\geq 0}\langle\tfrac32(1+2i+4j)\rangle+\langle\tfrac32(4+2i+4j)\rangle^*; && (7.17c)\\
\tilde\Delta_-(\tilde\Delta_-\otimes\{1^2\}) &= \sum_{i\geq 0,j\geq 0}\langle\tfrac32(3+2i+4j)\rangle+\langle\tfrac32(2+2i+4j)\rangle^*. && (7.17d)
\end{aligned}
$$

Since

$$
\sum_{i\geq 0,j\geq 0}\langle\tfrac32(a+2i+4j)\rangle=\sum_{m\geq a,m\equiv a \bmod 2}[\frac{m+4-a}{4}]\,\langle\tfrac32(m)\rangle \qquad (7.18)
$$

it then follows that

$$
\begin{aligned}
\tilde\Delta_+(\tilde\Delta_-\otimes\{2\}) &= \sum_{m\,\mathrm{even}}[\frac{m+2}{4}]\,\langle\tfrac32(m)\rangle+\sum_{m\,\mathrm{odd}}[\frac{m+1}{4}]\,\langle\tfrac32(m)\rangle^*; && (7.19a)\\
\tilde\Delta_+(\tilde\Delta_-\otimes\{1^2\}) &= \sum_{m\,\mathrm{even}}[\frac{m}{4}]\,\langle\tfrac32(m)\rangle+\sum_{m\,\mathrm{odd}}[\frac{m+3}{4}]\,\langle\tfrac32(m)\rangle^*; && (7.19b)\\
\tilde\Delta_-(\tilde\Delta_+\otimes\{2\}) &= \sum_{m\,\mathrm{odd}}[\frac{m+3}{4}]\,\langle\tfrac32(m)\rangle+\sum_{m\,\mathrm{even}}[\frac{m}{4}]\,\langle\tfrac32(m)\rangle^*; && (7.19c)\\
\tilde\Delta_-(\tilde\Delta_+\otimes\{1^2\}) &= \sum_{m\,\mathrm{even}}[\frac{m+1}{4}]\,\langle\tfrac32(m)\rangle+\sum_{m\,\mathrm{odd}}[\frac{m+2}{4}]\,\langle\tfrac32(m)\rangle^* && (7.19d)
\end{aligned}
$$

Combining the results (7.16), (7.17) and (7.19) and taking care to distinguish even and odd weight terms (7.16) we then have

$$\begin{aligned}
\tilde{\Delta}_+ \otimes \{3\} &= \sum_{m \mathrm{even}} (1+[\frac{m}{3}]-[\frac{m+2}{4}])\langle\tfrac{3}{2}(m)\rangle \\
&+ \sum_{m \mathrm{odd}} ([\frac{m}{3}]-[\frac{m+1}{4}])\langle\tfrac{3}{2}(m)\rangle^*; \qquad (7.20a)\\
\tilde{\Delta}_- \otimes \{3\} &= \sum_{m \mathrm{odd}} (1+[\frac{m}{3}]-[\frac{m+3}{4}])\langle\tfrac{3}{2}(m)\rangle \\
&+ \sum_{m \mathrm{even}} ([\frac{m}{3}]-[\frac{m}{4}])\langle\tfrac{3}{2}(m)\rangle^*; \qquad (7.20b)\\
\tilde{\Delta}_+ \otimes \{21\} &= \sum_{m \mathrm{even}} (m-[\frac{m}{3}]-[\frac{m}{4}]-[\frac{m+2}{4}])\langle\tfrac{3}{2}(m)\rangle \\
&+ \sum_{m \mathrm{odd}} (m-[\frac{m}{3}]-[\frac{m+1}{4}]-[\frac{m+3}{4}])\langle\tfrac{3}{2}(m)\rangle^*; \qquad (7.20c)\\
\tilde{\Delta}_- \otimes \{21\} &= \sum_{m \mathrm{odd}} (m-[\frac{m}{3}]-[\frac{m+1}{4}]-[\frac{m+3}{4}])\langle\tfrac{3}{2}(m)\rangle \\
&+ \sum_{m \mathrm{even}} (m-[\frac{m}{3}]-[\frac{m}{4}]-[\frac{m+2}{4}])\langle\tfrac{3}{2}(m)\rangle^*; \qquad (7.20d)\\
\tilde{\Delta}_+ \otimes \{1^3\} &= \sum_{m \mathrm{even}} ([\frac{m}{3}]-[\frac{m}{4}])\langle\tfrac{3}{2}(m)\rangle \\
&+ \sum_{m \mathrm{odd}} (1+[\frac{m}{3}]-[\frac{m+3}{4}])\langle\tfrac{3}{2}(m)\rangle^*; \qquad (7.20e)\\
\tilde{\Delta}_- \otimes \{1^3\} &= \sum_{m \mathrm{odd}} ([\frac{m}{3}]-[\frac{m+1}{4}])\langle\tfrac{3}{2}(m)\rangle \\
&+ \sum_{m \mathrm{even}} (1+[\frac{m}{3}]-[\frac{m+2}{4}])\langle\tfrac{3}{2}(m)\rangle^*. \qquad (7.20f)
\end{aligned}$$

Since

$$[\frac{(y+12x)}{3}]-[\frac{y+12x+a}{4}] = 1+[\frac{y}{3}]-[\frac{y+a}{4}] \qquad (7.21a)$$

and

$$\begin{aligned}
(y+12x)-[\frac{(y+12x)}{3}]-[\frac{y+12x+a}{4}]-[\frac{y+12x+b}{4}]& \\
= 2y+[\frac{y}{3}]-[\frac{y+a}{4}]-[\frac{y+b}{4}],& \qquad (7.21b)
\end{aligned}$$

for $0 \le y \le 11$ these results (7.20) can be rewritten in the form

$$\tilde{\Delta}_+ \otimes \{3\} = \sum_{x=0}\sum_{y=0}^{11}(m_y+x)\langle\tfrac{3}{2}(y+12x)\rangle^{(*)^y} \text{ with } m=(100010101110); \qquad (7.22a)$$

$$\tilde{\Delta}_- \otimes \{3\} = \sum_{x=0}^{11}\sum_{y=0}(m_y+x)\langle \tfrac{3}{2}(y+12x)\rangle^{(*)^{y+1}} \text{ with } m = (000100110111); \tag{7.22b}$$

$$\tilde{\Delta}_+ \otimes \{21\} = \sum_{x=0}^{11}\sum_{y=0}(m_y+2x)\langle \tfrac{3}{2}(y+12x)\rangle^{(*)^{y}} \text{ with } m = (001011112122); \tag{7.22c}$$

$$\tilde{\Delta}_- \otimes \{21\} = \sum_{x=0}^{11}\sum_{y=0}(m_y+2x)\langle \tfrac{3}{2}(y+12x)\rangle^{(*)^{y+1}} \text{ with } m = (001011112122); \tag{7.22d}$$

$$\tilde{\Delta}_+ \otimes \{1^3\} = \sum_{x=0}^{11}\sum_{y=0}(m_y+x)\langle \tfrac{3}{2}(y+12x)\rangle^{(*)^{y}} \text{ with } m = (000100110111); \tag{7.22e}$$

$$\tilde{\Delta}_- \otimes \{1^3\} = \sum_{x=0}^{11}\sum_{y=0}(m_y+x)\langle \tfrac{3}{2}(y+12x)\rangle^{(*)^{y+1}} \text{ with } m = (100010101110). \tag{7.22f}$$

where $(*)^z$ is to be ignored if z is even and set to be $*$ if z is odd.

Clearly, just as (7.4) is analogous to (7.3), so the results (7.22) for $Sp(2n,\Re)$ are analogous to the results (7.5) for $SO(2n)$. However, the analogy may not be quite what one might have expected. For ρ any partition of $k \leq 3$ the correspondence takes the form:

$$\Delta_+ \otimes \{\rho\} \iff \tilde{\Delta}_+ \otimes \{\rho\}; \tag{7.23a}$$
$$\Delta_- \otimes \{\rho\} \iff \tilde{\Delta}_- \otimes \{\rho'\}. \tag{7.23b}$$

To be more precise all our results support the validity of the following closing conjecture:

Conjecture 7.1 *Let ρ be an arbitrary partition of k and let t take values in the set $\{-1,0,1\}$. For $SO(2n)$ let*

$$\Delta_+ \otimes \{\rho\} = \begin{cases} \sum_{\lambda,t} p^{\rho}_{\lambda,t}\ [m^n/\lambda']_{\eta(t)} & \text{for } k=2m \text{ even;} \\ \sum_{\lambda,t} p^{\rho}_{\lambda,t}\ [\Delta; m^n/\lambda']_{\eta(t)} & \text{for } k=2m{+}1 \text{ odd,} \end{cases} \tag{7.24}$$

where if $k=2m$ and $\lambda'_1 = m$ then $t=0$ and $\eta(0)$ is to be omitted, while otherwise $t=\pm1$ with $\eta(1)=+$ and $\eta(-1)=-$. Similarly, for $Sp(2n,\Re)$ let

$$\tilde{\Delta}_+ \otimes \{\rho\} = \begin{cases} \sum_{\lambda,t} q^{\rho}_{\lambda,t}\ \langle \frac{1}{2}k(\lambda)\rangle^{\zeta(t)} & \text{for } k=2m \text{ even;} \\ \sum_{\lambda,t} q^{\rho}_{\lambda,t}\ \langle \frac{1}{2}k(\lambda)\rangle^{\zeta(t)} & \text{for } k=2m{+}1 \text{ odd,} \end{cases} \tag{7.25}$$

where if $\langle \frac{1}{2}k(\lambda)\rangle$ is self-associate so that $k=2m$ and $\lambda'_1=m$ then $t=0$ and $\zeta(0)$ is to be omitted, while otherwise $t=\pm1$ and $\zeta(1)$ is to be omitted while $\zeta(-1)$ is set to be $*$. Then

$$p^{\rho}_{\lambda,t} = q^{\rho}_{\lambda,t}. \tag{7.26}$$

It should be stressed that the non-zero terms of (7.24) are necessarily finite in number by virtue of the requirement that $\{m^n/\lambda'\}$ be non-vanishing. The same is not true of (7.25) which, as in (7.4) and (7.20), is expected to always involve an infinite number of terms.

While the corresponding formula for $\Delta_- \otimes \{\rho\}$ is obtained from (7.24) merely by replacing every surviving $\eta(\pm 1) = \pm$ by $\mp$, the corresponding formula for $\tilde{\Delta}_- \otimes \{\rho\}$ is obtained from (7.25) through the use of the conjugacy formula (5.6) of Proposition 5.2:

$$\tilde{\Delta}_- \otimes \{\rho\} = \left(\tilde{\Delta}_+ \otimes \{\rho'\}\right)^*. \tag{7.27}$$

This is well illustrated not only by (7.4) but also by (7.20).

8. Concluding remarks

In deriving the results obtained in this paper we have had two objectives in mind. Firstly, to gain further understanding of the properties of the unitary irreducible representations of the non-compact group $Sp(2n,\Re)$ and in particular their Kronecker products and plethysms. Secondly, to produce results and techniques aimed at eventual application in symplectic models of many-particle systems. The first objective has been achieved through an understanding, and proof, of hitherto conjectured properties of Kronecker products and plethysms of irreducible representations of $Sp(2n,\Re)$. That process has also generated a number of new identities involving plethysms of infinite series of S-functions. Progress with respect to the second objective has been advanced not only through the derivation of a highly efficient general formula for the evaluation of arbitrary plethysms, as well as specific results pertaining to symmetrised squares and cubes, but also through the introduction of associate irreducible representations of $Sp(2n,\Re)$ which allow one to compute Kronecker products and plethysms for particular irreducible representations and then to obtain additional results for the associate irreducible representations by a simple replacement process, at far less computational cost than that involved in repeating the entire calculations.

Acknowledgments

The work of BGW has been supported by a Polish KBN Grant. We appreciated the receipt of two prior independent proofs of Lemma 5.4 from Jean-Yves Thibon (Université de Marne-la-Vallée) and Thomas Scharf (Universität Bayreuth).

References

[1] Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)
[2] Rowe D J, Wybourne B G and Butler P H 1985 *J. Phys. A: Math. Gen.* **18** 939
[3] King R C and Wybourne B G 1985 *J. Phys. A: Math. Gen.* **18** 3113
[4] Rowe D J 1985 *Rep. Prog. Phys.* **48** 1419
[5] Haase R W and Johnson N F 1993 *Phys. Rev.* **B48** 1583

[6] Wybourne B G 1994 *Rept. Math. Phys.* **34** 9
[7] Kashiwara M and Vergne M 1978 *Inventiones Math.* **42** 1
[8] Littlewood D E 1940 *The Theory of Group Characters* (Oxford: Clarendon)
[9] Macdonald I G 1979 *Symmetric functions and Hall Polynomials* (Oxford: Clarendon)
[10] Carvalho M J 1990 *J. Phys. A: Math. Gen.* **23** 1909
[11] Wybourne B G 1992 *J. Phys. A: Math. Gen.* **25** 4389
[12] Grudzinski K and Wybourne B G 1995 in *Proc. Third International School on Theoretical Physics* Eds. Lulek, Florek and Wałcerz (Singapore: World Scientific) pp 469-483
[13] Grudzinski K and Wybourne B G 1996 *Rept. Math. Phys.* **38** 251
[14] Grudzinski K and Wybourne B G 1996 *J. Phys. A: Math. Gen.* **29** 6631
[15] Wybourne B G 1996 *Seminaire Lotharingien de Combinatoire* **B36**g
[16] Wybourne B G 1996 (Unpublished tables that are available at the WEB site http://www.phys.uni.torun.pl/~bgw/)
[17] Thibon J-Y, Toumazet F and Wybourne B G 1998 *J. Phys. A: Math. Gen.* **31** 1073
[18] Bargmann V 1954 *Ann. Math.* **59** 1
[19] Segal E 1959 *Mat. fys. Medd. Danske Vid. Selsk.* **31** 1
[20] Shale D 1962 *Trans. Amer. Math. Soc.* **103** 149
[21] Weil A 1964 *Acta Math.* **111** 143
[22] Moshinsky M and Quesne C 1971 *J. Math. Phys.* **12** 1772
[23] Howe R 1983 in *Representation Theory of Reductive Groups* Ed. Trombi (Boston: Birkhäuser) pp 159-175
[24] King R C 1975 *J. Phys. A: Math. Gen.* **8** 429
[25] Salam M A and Wybourne B G 1992 *J. Phys. A: Math. Gen.* **25** 2297
[26] Littlewood D E 1944 *Phil. Trans. Roy. Soc.* A**239** 305
[27] King R C, Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 2509