# Admissible partitions and the square of the Vandermonde determinant

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**Dedicated to the memory of Claude Itzykson (1938-1995)**

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C. ITZYKSON

March 1 1994

Dear Pr. Wybourne

We have just received today your very interesting contribution to the expansion of (powers of) the discriminant.

Indeed you seem to have discovered quite a striking phenomenon which deserves an explanation – Our preprint was thought too mathematical for Nucl. Phys. B, so it is resubmitted to Journ. Mod. Phys. A published in Singapore. I hope it will be found suitable there –

The subject seems to me to be still widely open = For instance is there a rule for the signs of the coefficients? What is the meaning of the vanishing terms you have found? What is their general feature? Is it significant that you found them starting at N = 8 ? etc...

Best wishes

Claude Itzykson

## Claude Itzykson's Letter

- Indeed you seem to have discovered a striking phenomenon which deserves an explanation ...
- The subject seems to me to be still widely open = For instance, is there a rule for the signs of the coefficients?
- What is the meaning of the vanishing terms you have found?
- What is their general feature?
- Is it significant that you found them starting at $N = 8$? etc..
- Claude Itzykson March 1 1994

## Expansion of the Laughlin wavefunction

- Laughlin (1983) described the fractional quantum Hall effect in terms of a wavefunction

$$\Psi^m_{Laughlin}(z_1,\ldots,z_N) = \prod_{i<j}^{N}(z_i - z_j)^{2m+1}\exp\left(-\frac{1}{2}\sum_{i=1}^{N}|z_i|^2\right) \quad (1)$$

The Vandermonde alternating function in $N$ variables is defined as

$$V(z_1,\ldots,z_N) = \prod_{i<j}^{N}(z_i - z_j) \quad (2)$$

$$\frac{\Psi_{Laughlin}}{V} = V^{2m} = \sum_{\lambda\vdash n} c^\lambda s_\lambda \quad (3)$$

where $n = mN(N-1)$ and the $s_\lambda$ are Schur functions.

## Coefficients of the expansion

- The coefficients $c_\lambda$ are signed integers. Henceforth we consider the case where $m = 1$. The partitions, $(\lambda)$, indexing the Schur functions are of weight $N(N-1)$. Algorithms exist for computing the expansions and complete results have been obtained for $N \leq 10$. For a given $N$ the partitions are bounded by a highest partition $(2N-2, 2N-4, \ldots, 0)$ and a lowest partition $((N-1)^{N-1})$ with the partitions being of length $N$ and $N-1$. Let

$$n_k = \sum_{i=0}^{k} \lambda_{N-i} - k(k+1)k = 0, 1, \ldots, N-1 \tag{4}$$

## Admissible Partitions

- Di Francesco *et al* define *admissible partitions* as satisfying Eq(4) with *all* $n_k \geq 0$. They computed the number of admissible partitions $A_N$ for $N \leq 29$ and conjectured that $A_N$ was the number of distinct partitions arising in the expansion, Eq(3), *provided none of the coefficients vanished.*
- The conjecture fails for $N \geq 8$. We find the number of admissible partitions associated with vanishing coefficients as

$$(N = 8) \quad 8, \ (N = 9) \quad 66, \ (N = 10) \quad 389$$

## Admissible partitions whose coefficients vanish

- **Reversed partition symmetry**
- The coefficients of $s_\lambda$ and $s_{\lambda_r}$ are equal if

$$(\lambda_r) = (2(N-1) - \lambda_N, \ldots, 2(N-1) - \lambda_1) \tag{5}$$

- We list the 8 partitions for $N = 8$ as reverse pairs

$$\{13\ 11\ 985^2 41\} \quad \{13\ 10\ 9^2 6531\} \qquad (Q1)$$

$$\{13\ 11\ 9854^2 2\} \quad \{13\ 10\ 987531\} \qquad (Q2)$$

$$\{13\ 11\ 976541\} \quad \{12\ 10^2 96531\} \qquad (Q3)$$

$$\{12\ 11\ 97^2 4^2 1\} \quad \{12\ 10^2 7^2 532\} \qquad (Q4)$$

## The $q$–discriminant

- Let $q\mathbf{x} = (qx_1, qx_2, \ldots, qx_N)$ and the $q$–discriminant of $\mathbf{x}$ be

$$D_N(q;\mathbf{x}) = \prod_{1\le i\ne j\le N} (x_i - qx_j) \tag{6}$$

and

$$R_N(q;\mathbf{x}) = \prod_{1\le i\ne j\le N} (x_i - qx_j)(qx_i - x_j) = \sum_\lambda c^\lambda(q)s_\lambda(\mathbf{x}) \tag{7}$$

So that

$$V_N^2(\mathbf{x}) = \prod_{1\le i<j\le N} (x_i - x_j)^2 = R_N(1;\mathbf{x}) \tag{8}$$

Introduce $q$–polynomials such that

$$R_N(q;\mathbf{x}) = \sum_\lambda c^\lambda(q)s_\lambda(\mathbf{x}) \tag{9}$$

## The $q$–polynomials

$$R_N(q;\mathbf{x}) \quad = \frac{(-1)^{N(N-1)/2}}{(1-q)^N} \sum_{\nu\subseteq(N-1)^N} \left((-q)^{|\nu|)} + (-q)^{N^2-|\nu|}\right) \times s_{(N-1)^N/\nu}(\mathbf{x}) s_{\nu'}(\mathbf{x})$$

Such expansions have been evaluated as polynomials in $q$ for all admissible partitions for $N = 2...6$ with many examples for $N = 7, 8, 9$.

## Some $q-$polynomials

| | | | |
|---|---|---|---|
| N=2 | [1] | $q$ | $\{2\}$ |
| | [-3] | $-(q^2+q+1)$ | $\{1^2\}$ |
| N=3 | [1] | $q^3$ | $\{42\}$ |
| | [-3] | $-q^2(q^2+q+1)$ | $\{41^2\}+\{3^2\}$ |
| | [6] | $+q(q^2+q+1)(q^2+1)$ | $\{321\}$ |
| | [-15] | $-(q^2+q+1)(q^4+q^2+q+1)$ | $\{2^3\}$ |
| N=4 | [1] | $q^6$ | $\{642\}$ |
| | [-3] | $-q^5(q^2+q+1)$ | $\{641^2\}+\{63^2\}+\{5^22\}$ |
| | [6] | $+q^4(q^2+q+1)(q^2+1)$ | $\{6321\}+\{543\}$ |
| | .. | ..... | .... |

## $N = 8$ $q-$**polynomials where** $c_\lambda(1) = 0$

The $q-$polynomials for the four pairs of partitions designated earlier as $Q(1)...Q(4)$ are

$$\begin{array}{ll} Q(1) & -q^{17}(q^2-q+1)^2(q^2+1)^2(q^2+q+1)^5(1-q)^4 \\ Q(2) & +q^{16}(q^2-q+1)^2(q^2+1)(q^2+q+1)^6(1-q)^4 \\ Q(3) & +q^{16}(q^2-q+1)^2(q^2+1)^3(q^2+q+1)^5(1-q)^4 \\ Q(4) & +q^{14}(q^2-q+1)^2(q^2+q+1)^5(1-q)^4 \\ & \times(q^{10}+q^9+3q^8+4q^6+q^5+4q^4+3q^2+q+1) \end{array}$$

Note the factor $(q-1)^4$.

## Some Conjectures

- If a $q-$polynomial is of the form $(-1)^{\phi} q^{p} Q(q)$ then under $N \to N+1$

$$\phi \to \phi,\ \ p \to p+N,\ \ Q(q) \to Q(q),\ \ \{\lambda\} \to \{2N-2, \lambda\}$$

- Define

$$QS(N) = \sum_{\lambda} c_{\lambda}(q)$$

then

$$QS(N) = \prod_{x=0}^{[N/2]} (-3x+1) \prod_{x=0}^{[(N-1)/2]} (6x+1)$$

## Sum of Squares Problem

- Di Francesco etal establish the remarkable result that the sum of the squares of the coefficients of the second power of the Vandermonde with $q = 1$ is

$$\frac{(3N)!}{N!(3!)^N}$$

What is the corresponding result for the $q-$polynomials?

- For $N = 4$ one finds

$$q^{24} + 6q^{23} + 22q^{22} + 58q^{21} + 128q^{20} + 242q^{19}$$
$$+418q^{18} + 646q^{17} + 929q^{16} + 1210q^{15} + 1490q^{14}$$
$$+1670q^{13} + 1760q^{12} + 1670q^{11} + 1490q^{10} + 1210q^{9}$$
$$+646q^{8} + 418q^{6} + 242q^{5} + 128q^{4} + 58q^{3} + 22q^{2} + 6q + 1$$

Note the polynomial is symmetrical and unimodal!

## References


- G V Dunne, Slater Decomposition of Laughlin States, *Int. J. Mod. Phys.* **B7**,4783 (1993)
- P Di Francesco, M Gaudin, C Itzykson and F Lesage, Laughlin's wave functions, Coulomb gases and expansions of the discriminant *Int. J. Mod. Phys.* **A9**,4257 (1994)
- T Scharf, J-Y Thibon and B G Wybourne, Powers of the Vandermonde determinant and the quantum Hall effect *J Phys A:Math. Gen.* **27**, 4211 (1994)


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- The Vandermonde alternating function in $N$ variables is defined as

$$V(z_1,\ldots,z_N)=\prod_{i<j}^{N}(z_i-z_j)$$

$$\frac{\Psi_{Laughlin}}{V}=V^{2m}=\sum_{\lambda\vdash n}c^{\lambda}s_{\lambda}$$

where $n=mN(N-1)$ and the $s_\lambda$ are Schur functions.

## Principal Topics Considered

- Determination of the signed integers $c_\lambda$ $N = 2, \ldots, 10$
- Admissible partitions and zero coefficients
- The $q-$discriminant and $q-$polynomials $c_\lambda(q)$
- A conjecture

$$QS(N) = \sum_\lambda c_\lambda(q) = \prod_{x=0}^{[N/2]} (-3x+1) \prod_{x=0}^{[(N-1)/2]} (6x+1)$$

- Sum of squares - the $q-$polynomial is symmetrical and unimodal