SCHUR based Conjectures

One of the most useful features of SCHUR is the establishment of conjectures. From time to time I will place here new conjectures with the hope that someone will prove or disprove them.

The KW Plethysm Conjecture

In the notation of Littlewood Suppose $\{\lambda\}$ denotes a non-trivial $S-$function associated with the partition (λ) the we conjecture that:-

If $\{\lambda\}^p$ contains the $S-$function $\{\mu\}$ $n-$times and $p >= 3$ then if

$$\{\lambda\} \otimes \{p\} \supset s\{\mu\}$$
$$\{\lambda\} \otimes \{1^p\} \supset a\{\mu\}$$

then $a, s < n$.

Conjecture 2

Let D be the infinite series of $S-$functions having all parts even. Let D_k be those members of the D series with not more than k parts. We conjecture that:-

$$D_k \otimes \{2\} = \sum_{i=0}^{\infty} \{D_k^{4i} \cdot D_k\} \tag{1}$$
$$D_k \otimes \{1^2\} = \sum_{i=0}^{\infty} \{D_k^{4i+2} \cdot D_k\} \tag{2}$$

Solution to Conjecture 2

This follows by noting that

$$D = \{2\} \otimes M$$
$$\sum_{i=0}^{\infty} D^{4i} = \{2\} \otimes M_+$$
$$D \otimes \{2\} = \{2\} \otimes M \otimes \{2\}$$

But $M \otimes \{2\} = M\dot{M}_+$ and hence

$$D \otimes \{2\} = \{2\} \otimes M\dot{M}_+$$

But

$$\sum_{i=0}^{\infty} D^{4i}\dot{D} = \{2\} \otimes M_+\dot{\{2\}} \otimes M$$
$$= \{2\} \otimes M\dot{M}_+$$

and hence we are led to

$$D_k \otimes \{2\} = \sum_{i=0}^{\infty} \{D_k^{4i} \cdot D_k\}$$

The second part of the conjecture follows in the same manner after noting that

$$M \otimes \{1^2\} = M\dot{M}_-$$

Conjecture 3 Kronecker Products in $Sp(2n, R)$

1. The harmonic discrete series irreps of $Sp(2n, R)$ may be uniquely labelled as $< \frac{k}{2}; (\lambda) >$ where the partitions (λ) are restricted by the requirement that

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq k \tag{1}$$
$$\tilde{\lambda}_1 \leq n \tag{2}$$

2. Eqs.(1) and (2) restrict the length ℓ_λ of the partition (λ). If $n \geq \ell_\lambda > k$ then the partition is non-standard in $O(k)$. The modification rules for $O(k)$ give

$$[\lambda] = (-1)^{x-1}[\lambda - h]^* \text{where } h = 2\ell_\lambda - k \tag{3}$$

where x is the last column used in the removal of the continuous rim hook of length h. Given the restrictions of Eqs. (1) and (2) we are limited to $x = 1$ and removing $k - \tilde{\lambda}_1$ boxes from the first column to leave an associated irrep of $O(k)$. Conversely, starting with a standard irrep $[\lambda]$ of $O(k)$ we may form a non-standard irrep of $O(k)$ by adding $h_\lambda = k - 2\ell_\lambda$ boxes to the foot of the first column of (λ). This will give a standard labelled irrep of $Sp(2n, R)$ though the partition $(\lambda + h_\lambda)$ is $O(k)$ non-standard. For brievity we shall write

$$[\lambda + h_\lambda] \equiv [\lambda_h] \tag{4}$$

Under $O(k)$ standardisation

$$[\lambda_h] \to [\lambda]^* \tag{5}$$

.

3. Let us now define

$$< \frac{k}{2}; (\lambda) >^* = < \frac{k}{2}; (\lambda)^* > \tag{6}$$

4. From KW (8.15) (This refers to publication 95) we have

$$< \frac{k}{2}; (\mu) > \times < \frac{\ell}{2}; (\nu) > = \sum_\lambda R^{\mu\nu}_\lambda < \frac{k+\ell}{2}; (\lambda) > \tag{7}$$

where the coefficients $R^{\mu\nu}_\lambda$ are the branching rule coefficients appropriate to the restriction $O(k+\ell) \to O(k) \times O(\ell)$. Since $(R^{\mu\nu}_\lambda)^* = R^{\mu\nu}_\lambda$ we have

$$(< \frac{k}{2}; (\mu) > \times < \frac{\ell}{2}; (\nu) >)^* = < \frac{k}{2}; (\mu)^* > \times < \frac{\ell}{2}; (\nu)^* > \tag{8}$$

5. Let us define operators $\mathcal{M}$ and $\mathcal{M}^{-1}$ such that

$$\mathcal{M} < \frac{k}{2}; (\lambda_h) > = < \frac{k}{2}; (\lambda) > \tag{8a}$$

$$\mathcal{M}^{-1} < \frac{k}{2}; (\lambda) > = < \frac{k}{2}; (\lambda_h) > \tag{8b}$$

Conjectures on Kronecker products in $Sp(2n, R)$

If $< \frac{k}{2}; (\mu) > = (\mathcal{M}$ or $\mathcal{M}^{-1})(< \frac{k}{2}; (\mu') >)$ and likewise for $< \frac{\ell}{2}; (\nu) >$ then for sufficiently large n

$$\mathcal{M}(< \frac{k}{2}; (\mu) >< \frac{\ell}{2}; (\nu) >) = \mathcal{M}(< \frac{k}{2}; (\mu') >< \frac{\ell}{2}; (\nu') >) \tag{9a}$$

$$\mathcal{M}^{-1}(< \frac{k}{2}; (\mu) >< \frac{\ell}{2}; (\nu) >) = \mathcal{M}^{-1}(< \frac{k}{2}; (\mu') >< \frac{\ell}{2}; (\nu') >) \tag{9b}$$

By sufficiently large n we mean large enough so that no $Sp(2n, R)$ modification rules are needed. In every case it is to be understood that $\mathcal{M}$ or $\mathcal{M}^{-1}$ acts on the resultant of the Kronecker product.

In certain cases the action of $\mathcal{M}$ or $\mathcal{M}^{-1}$ on one side or the other will have no effect and the conjecture simplifies.

Conjectures on Kronecker Powers in $Sp(2n, R)$

Considerations of the squares of irreps of $Sp(2n, R)$ strongly suggest the following conjecture:-

$$M(< \frac{k}{2}; (1^k) >^2) = < \frac{k}{2}; (0) >^2 \tag{10}$$

Limited data on cubes would suggest that Eq.(10) generalises to

$$M(< \frac{k}{2}; (1^k) >^p) = M(< \frac{k}{2}; (0) >^p) \tag{11}$$

and further that

$$M(<\frac{k}{2};(\lambda_h)>^p) = M(<\frac{k}{2};(\lambda)>^p) \tag{12}$$

Some examples for $Sp(18,R)$

In these examples we use $\mathcal{M}$ though they are equally valid for $\mathcal{M}^{-1}$. If $\mathcal{M}$ is on the left-hand-side only then the other case has $\mathcal{M}^{-1}$ on the right-hand-side.

$$\mathcal{M}(<s;(0)><1;(1)>) = \mathcal{M}(<s;(1)><1;(1)>) \tag{13a}$$
$$\mathcal{M}(<s;(0)><1;(1^2)>) = <s;(1)><1;(0)> \tag{13b}$$
$$\mathcal{M}(<1;(1^2)><1;(1^2)>) = <1;(0)><1;(0)> \tag{13c}$$
$$\mathcal{M}(<1;(1)><1;(1^2)>) = <1;(1)><1;(0)> \tag{13d}$$
$$\mathcal{M}(<s1;(31)><s1;(21)>) = \mathcal{M}(<s1;(3)><s1;(21)>) \tag{13c}$$
$$\mathcal{M}(<2;(21^2)><1;(0)>) = <2;(2)><1;(1^2)> \tag{13d}$$
$$\mathcal{M}(<2;(31^2)><2;(2)>) = \mathcal{M}(<2;(3)><2;(21^2)>) \tag{13e}$$
$$\mathcal{M}(<2;(31^2)><2;(21^2)>) = \mathcal{M}(<2;(3)><2;(2)>) \tag{13f}$$
$$\mathcal{M}(<2;(21^2)><2;(21^2)>) = \mathcal{M}(<2;(2)><2;(2)> \tag{13g}$$
$$\mathcal{M}(<2;(1^4)><2;(1^4)>) = <2;(0)><2;(0)> \tag{13h}$$
$$\mathcal{M}(<3;(1^4)><3;(1^4)>) = <3;(1^2)><3;(1^2)> \tag{13i}$$

Conjectures on Reduced Inner Plethysms

The relevant references are contained in papers 84 and 132 of Publications.

Suppose X stands for one of the infinite S-fn series designated in 84 as A, C, E, G, H, L, P, R, W. The coefficients in these series carry a phase of ± 1. Then we make the following conjectures concerning the coefficients c^ρ in the expansion of

$$< 1 > \otimes \{\lambda / X\} = \sum_\rho c^\rho < \rho >$$

$$c^\rho >= 0 \quad all\ \lambda \quad for\ X = C, G, L, V \tag{1a}$$

$$c^\rho >= 0 \quad all\ \lambda \supset T \quad for\ X = A, E, P, W \tag{1b}$$

For $X = H, R$ the c^ρ may be ± 1.

The case for $X = G$ is obvious since $< 1 > \otimes \{\lambda / G\}$ enumerates the group subgroup decomposition $O(n) - > S(n)$. Likewise $< 1 > \otimes \{\lambda / C\}$ enumerates the group subgroup decomposition $O(n-1) - > S(n)$ but what do the other examples enumerate?

A Plethysm Conjecture for the $M-$Series

Consider the infinite $S-$function series

$$M = \sum_{m=0}^{\infty} \{m\} \tag{1}$$

with

$$M_{+} = \sum_{m=0}^{\infty} \{2m\} \tag{2a}$$

$$M_{-} = \sum_{m=0}^{\infty} \{2m+1\} \tag{2b}$$

We conjecture that if (λ) is a partition of weight w_λ and $(\tilde{\lambda})$ is the partition conjugate to (λ) then we conjecture that

$$M_{+} \otimes \{\lambda\} \ \supset (M_{-} \otimes \{\tilde{\lambda}\})/\{1^{w_\lambda}\} \tag{3a}$$

$$M_{-} \otimes \{\tilde{\lambda}\} \ \supset (M_{+} \otimes \{\lambda\})/\{1^{w_\lambda}\} \tag{3b}$$

The above conjecture would imply that it is suffiсent to compute only the w_λ parts of the two plethysms to be then able to deduce all terms involving partitions of fewer than w_λ parts.

Supporting evidence for this conjecture may be seen in the following tables:-

$$M_{+} \otimes \{3\} =$$

$\{0\}$	$+\ \{2\}$	$+\ \{2^2\}$	$+\ \{2^3\}$	$+\ 2\{4\}$	$+\ 2\{42\}$	$+\ \{42^2\}$
$+\ 2\{4^2\}$	$+\ \{4^2 2\}$	$+\ \{4^3\}$	$+\ \{51\}$	$+\ \{521\}$	$+\ \{53\}$	$+\ \{541\}$
$+\ 3\{6\}$	$+\ 4\{62\}$	$+\ 2\{62^2\}$	$+\ \{631\}$	$+\ 4\{64\}$	$+\ 2\{642\}$	$+\ \{651\}$
$+\ 3\{6^2\}$	$+\ 2\{71\}$	$+\ 2\{721\}$	$+\ 3\{73\}$	$+\ \{732\}$	$+\ 3\{741\}$	$+\ 2\{75\}$
$+\ 4\{8\}$	$+\ 6\{82\}$	$+\ 3\{82^2\}$	$+\ 2\{831\}$	$+\ 7\{84\}$	$+\ 4\{91\}$	$+\ 3\{921\}$
$+\ 6\{93\}$	$+\ 5\{10\ \}$	$+\ \{10\ 1^2\}$	$+\ 9\{10\ 2\}$	$+\ 5\{11\ 1\}$	$+\ 7\{12\ \}$	

$$M_{-} \otimes \{1^3\} =$$

$\{1^3\}$	$+\ \{31^2\}$	$+\ \{3^2 1\}$	$+\ \{3^3\}$	$+\ \{41\}$	$+\ \{43\}$	$+\ 2\{51^2\}$
$+\ \{52\}$	$+\ 2\{531\}$	$+\ \{53^2\}$	$+\ \{54\}$	$+\ 2\{5^2 1\}$	$+\ \{5^2 3\}$	$+\ 2\{61\}$
$+\ \{621\}$	$+\ 3\{63\}$	$+\ \{632\}$	$+\ \{641\}$	$+\ 2\{65\}$	$+\ \{652\}$	$+\ 3\{71^2\}$
$+\ 2\{72\}$	$+\ 4\{731\}$	$+\ 2\{73^2\}$	$+\ 3\{74\}$	$+\ \{742\}$	$+\ 4\{751\}$	$+\ 2\{76\}$
$+\ 3\{81\}$	$+\ 2\{821\}$	$+\ 5\{83\}$	$+\ 2\{832\}$	$+\ 3\{841\}$	$+\ 5\{85\}$	$+\ \{9\}$
$+\ 4\{91^2\}$	$+\ 4\{92\}$	$+\ 6\{931\}$	$+\ 6\{94\}$	$+\ 5\{10\ 1\}$	$+\ 4\{10\ 21\}$	$+\ 8\{10\ 3\}$
$+\ \{11\ \}$	$+\ 5\{11\ 1^2\}$	$+\ 6\{11\ 2\}$	$+\ 7\{12\ 1\}$	$+\ 2\{13\ \}$		

$$M_+ \otimes \{21\} =$$

$\{2\}$	$+\ \{2^2\}$	$+\ \{31\}$	$+\ \{321\}$	$+\ 2\{4\}$	$+\ 3\{42\}$	$+\ \{42^2\}$
$+\ \{431\}$	$+\ 2\{4^2\}$	$+\ \{4^22\}$	$+\ 3\{51\}$	$+\ 2\{521\}$	$+\ 3\{53\}$	$+\ \{532\}$
$+\ 2\{541\}$	$+\ \{543\}$	$+\ \{5^2\}$	$+\ 3\{6\}$	$+\ \{61^2\}$	$+\ 6\{62\}$	$+\ 2\{62^2\}$
$+\ 3\{631\}$	$+\ 6\{64\}$	$+\ 3\{642\}$	$+\ 3\{651\}$	$+\ 3\{6^2\}$	$+\ 5\{71\}$	$+\ 4\{721\}$
$+\ 7\{73\}$	$+\ 3\{732\}$	$+\ 5\{741\}$	$+\ 6\{75\}$	$+\ 5\{8\}$	$+\ 2\{81^2\}$	$+\ 10\{82\}$
$+\ 3\{82^2\}$	$+\ 6\{831\}$	$+\ 12\{84\}$	$+\ 8\{91\}$	$+\ 7\{921\}$	$+\ 12\{93\}$	$+\ 7\{10\ \}$
$+\ 3\{10\ 1^2\}$	$+\ 15\{10\ 2\}$	$+\ 12\{11\ 1\}$	$+\ 9\{12\ \}$			

$$M_- \otimes \{21\} =$$

$\{21\}$	$+\ \{31^2\}$	$+\ \{32\}$	$+\ \{3^21\}$	$+\ 2\{41\}$	$+\ \{421\}$	$+\ 2\{43\}$
$+\ \{432\}$	$+\ \{5\}$	$+\ 2\{51^2\}$	$+\ 3\{52\}$	$+\ 3\{531\}$	$+\ \{53^2\}$	$+\ 3\{54\}$
$+\ \{542\}$	$+\ 2\{5^21\}$	$+\ \{5^23\}$	$+\ 4\{61\}$	$+\ 3\{621\}$	$+\ 5\{63\}$	$+\ 2\{632\}$
$+\ 3\{641\}$	$+\ \{643\}$	$+\ 4\{65\}$	$+\ 2\{652\}$	$+\ \{6^21\}$	$+\ 2\{7\}$	$+\ 3\{71^2\}$
$+\ 6\{72\}$	$+\ \{72^2\}$	$+\ 6\{731\}$	$+\ 2\{73^2\}$	$+\ 7\{74\}$	$+\ 3\{742\}$	$+\ 6\{751\}$
$+\ 5\{76\}$	$+\ 7\{81\}$	$+\ 5\{821\}$	$+\ 10\{83\}$	$+\ 4\{832\}$	$+\ 7\{841\}$	$+\ 10\{85\}$
$+\ 3\{9\}$	$+\ 5\{91^2\}$	$+\ 10\{92\}$	$+\ 2\{92^2\}$	$+\ 10\{931\}$	$+\ 13\{94\}$	$+\ 10\{10\ 1\}$
$+\ 8\{10\ 21\}$	$+\ 16\{10\ 3\}$	$+\ 5\{11\ \}$	$+\ 7\{11\ 1^2\}$	$+\ 15\{11\ 2\}$	$+\ 14\{12\ 1\}$	$+\ 7\{13\ \}$

$$M_+ \otimes \{4\} =$$

$\{0\}$	$+\ \{2\}$	$+\ \{2^2\}$	$+\ \{2^3\}$	$+\ \{2^4\}$	$+\ 2\{4\}$	$+\ 2\{42\}$
$+\ 2\{42^2\}$	$+\ \{42^3\}$	$+\ 3\{4^2\}$	$+\ 2\{4^22\}$	$+\ \{4^22^2\}$	$+\ 2\{4^3\}$	$+\ \{51\}$
$+\ \{521\}$	$+\ \{52^21\}$	$+\ \{53\}$	$+\ \{532\}$	$+\ 2\{541\}$	$+\ \{5421\}$	$+\ \{543\}$
$+\ \{5^21^2\}$	$+\ 3\{6\}$	$+\ 5\{62\}$	$+\ 4\{62^2\}$	$+\ 2\{62^3\}$	$+\ 2\{631\}$	$+\ \{6321\}$
$+\ 6\{64\}$	$+\ 7\{642\}$	$+\ 3\{651\}$	$+\ 6\{6^2\}$	$+\ 2\{71\}$	$+\ 3\{721\}$	$+\ 2\{72^21\}$
$+\ 5\{73\}$	$+\ \{731^2\}$	$+\ 4\{732\}$	$+\ 7\{741\}$	$+\ 4\{75\}$	$+\ 5\{8\}$	$+\ 8\{82\}$
$+\ 8\{82^2\}$	$+\ 5\{831\}$	$+\ 13\{84\}$	$+\ 5\{91\}$	$+\ 6\{921\}$	$+\ 10\{93\}$	$+\ 6\{10\ \}$
$+\ \{10\ 1^2\}$	$+\ 14\{10\ 2\}$	$+\ 7\{11\ 1\}$	$+\ 9\{12\ \}$			

$$M_- \otimes \{1^4\} =$$

$\{1^4\}$	$+\ \{31^3\}$	$+\ \{3^21^2\}$	$+\ \{3^31\}$	$+\ \{3^4\}$	$+\ \{41^2\}$
$+\ \{431\}$	$+\ \{43^2\}$	$+\ \{4^2\}$	$+\ 2\{51^3\}$	$+\ \{521\}$	$+\ 2\{531^2\}$
$+\ \{532\}$	$+\ 2\{53^21\}$	$+\ \{53^3\}$	$+\ 2\{541\}$	$+\ \{543\}$	$+\ 3\{5^21^2\}$
$+\ \{5^22\}$	$+\ 2\{5^231\}$	$+\ \{5^24\}$	$+\ 2\{61^2\}$	$+\ \{62\}$	$+\ \{621^2\}$
$+\ 4\{631\}$	$+\ \{6321\}$	$+\ 3\{63^2\}$	$+\ \{63^22\}$	$+\ 2\{64\}$	$+\ \{641^2\}$
$+\ 3\{642\}$	$+\ \{6431\}$	$+\ 4\{651\}$	$+\ 2\{6521\}$	$+\ 4\{653\}$	$+\ 3\{6^2\}$
$+\ 2\{6^22\}$	$+\ 3\{71^3\}$	$+\ 3\{721\}$	$+\ 2\{73\}$	$+\ 5\{731^2\}$	$+\ 3\{732\}$
$+\ 4\{73^21\}$	$+\ 7\{741\}$	$+\ 2\{7421\}$	$+\ 5\{743\}$	$+\ 2\{75\}$	$+\ 6\{751^2\}$
$+\ 7\{752\}$	$+\ 7\{761\}$	$+\ \{7^2\}$	$+\ 4\{81^2\}$	$+\ \{82\}$	$+\ 2\{821^2\}$
$+\ \{82^2\}$	$+\ 9\{831\}$	$+\ 3\{8321\}$	$+\ 7\{83^2\}$	$+\ 5\{84\}$	$+\ 5\{841^2\}$
$+\ 7\{842\}$	$+\ 13\{851\}$	$+\ 6\{86\}$	$+\ \{91\}$	$+\ 5\{91^3\}$	$+\ 6\{921\}$
$+\ 4\{93\}$	$+\ 8\{931^2\}$	$+\ 8\{932\}$	$+\ 15\{941\}$	$+\ 8\{95\}$	$+\ 6\{10\ 1^2\}$
$+\ 4\{10\ 2\}$	$+\ 5\{10\ 21^2\}$	$+\ 2\{10\ 2^2\}$	$+\ 17\{10\ 31\}$	$+\ 10\{10\ 4\}$	$+\ 2\{11\ 1\}$
$+\ 6\{11\ 1^3\}$	$+\ 11\{11\ 21\}$	$+\ 9\{11\ 3\}$	$+\ 10\{12\ 1^2\}$	$+\ 6\{12\ 2\}$	$+\ 4\{13\ 1\}$

$$M_+ \otimes \{31\} =$$

$\{2\} + \{2^2\} + \{2^3\} + \{31\} + \{321\} + \{32^21\}$

$+ 2\{4\} + 4\{42\} + 3\{42^2\} + \{42^3\} + 2\{431\} + \{4321\}$

$+ 3\{4^2\} + 4\{4^22\} + \{4^22^2\} + \{4^231\} + 2\{4^3\} + \{4^32\}$

$+ 3\{51\} + 4\{521\} + 2\{52^21\} + 5\{53\} + \{531^2\} + 4\{532\}$

$+ \{532^2\} + 6\{541\} + 3\{5421\} + 4\{543\} + \{5432\} + 2\{54^21\}$

$+ 2\{5^2\} + \{5^21^2\} + 3\{5^22\} + \{5^231\} + \{5^24\} + 4\{6\}$

$+ \{61^2\} + 9\{62\} + \{621^2\} + 8\{62^2\} + 2\{62^3\} + 8\{631\}$

$+ 4\{6321\} + 2\{63^2\} + 13\{64\} + 3\{641^2\} + 15\{642\} + 4\{642^2\}$

$+ 4\{6431\} + 8\{64^2\} + 11\{651\} + 6\{6521\} + 8\{653\} + 8\{6^2\}$

$+ 2\{6^21^2\} + 13\{6^22\} + 7\{71\} + 10\{721\} + 5\{72^21\} + 13\{73\}$

$+ 3\{731^2\} + 14\{732\} + 3\{732^2\} + 2\{73^21\} + 20\{741\} + 11\{7421\}$

$+ 15\{743\} + 16\{75\} + 7\{751^2\} + 22\{752\} + 20\{761\} + 6\{7^2\}$

$+ 6\{8\} + 3\{81^2\} + 18\{82\} + 3\{821^2\} + 15\{82^2\} + 4\{82^3\}$

$+ 20\{831\} + 10\{8321\} + 7\{83^2\} + 28\{84\} + 8\{841^2\} + 38\{842\}$

$+ 34\{851\} + 31\{86\} + 12\{91\} + 20\{921\} + 9\{92^21\} + 28\{93\}$

$+ 8\{931^2\} + 30\{932\} + 45\{941\} + 39\{95\} + 10\{10\;\} + 7\{10\;1^2\}$

$+ 30\{10\;2\} + 6\{10\;21^2\} + 27\{10\;2^2\} + 40\{10\;31\} + 54\{10\;4\} + 20\{11\;1\}$

$+ \{11\;1^3\} + 35\{11\;21\} + 48\{11\;3\} + 14\{12\;\} + 12\{12\;1^2\} + 48\{12\;2\}$

$+ 30\{13\;1\} + 20\{14\;\}$

$$M_- \otimes \{21^2\} =$$

$\{21^2\} + \{31^3\} + \{321\} + \{3^21^2\} + \{3^22\} + \{3^31\}$

$+ 2\{41^2\} + \{42\} + \{421^2\} + 3\{431\} + \{4321\} + 2\{43^2\}$

$+ \{43^22\} + \{4^2\} + \{4^22\} + \{51\} + 2\{51^3\} + 4\{521\}$

$+ 3\{53\} + 4\{531^2\} + 4\{532\} + 3\{53^21\} + \{53^3\} + 6\{541\}$

$+ 2\{5421\} + 4\{543\} + \{5432\} + 2\{5^2\} + 3\{5^21^2\} + 5\{5^22\}$

$+ 4\{5^231\} + 3\{5^24\} + 5\{61^2\} + 3\{62\} + 3\{621^2\} + 2\{62^2\}$

$+ 11\{631\} + 4\{6321\} + 7\{63^2\} + 2\{63^22\} + 7\{64\} + 5\{641^2\}$

$+ 8\{642\} + \{642^2\} + 4\{6431\} + 2\{64^2\} + 13\{651\} + 6\{6521\}$

$+ 11\{653\} + 4\{6^2\} + 2\{6^21^2\} + 7\{6^22\} + 3\{71\} + 4\{71^3\}$

$+ 10\{721\} + \{72^21\} + 8\{73\} + 9\{731^2\} + 13\{732\} + \{732^2\}$

$+ 8\{73^21\} + 20\{741\} + 8\{7421\} + 15\{743\} + 12\{75\} + 13\{751^2\}$

$+ 22\{752\} + 20\{761\} + 6\{7^2\} + 9\{81^2\} + 8\{82\} + 7\{821^2\}$

$+ 5\{82^2\} + 25\{831\} + 10\{8321\} + 16\{83^2\} + 17\{84\} + 13\{841^2\}$

$+ 26\{842\} + 38\{851\} + 21\{86\} + 6\{91\} + 6\{91^3\} + 20\{921\}$

$+ 3\{92^21\} + 19\{93\} + 18\{931^2\} + 27\{932\} + 45\{941\} + 30\{95\}$

$+ \{10\;\} + 16\{10\;1^2\} + 15\{10\;2\} + 12\{10\;21^2\} + 12\{10\;2^2\} + 48\{10\;31\}$

$+ 36\{10\;4\} + 11\{11\;1\} + 10\{11\;1^3\} + 35\{11\;21\} + 34\{11\;3\} + 2\{12\;\}$

$+ 24\{12\;1^2\} + 27\{12\;2\} + 18\{13\;1\} + 4\{14\;\}$

$$M_+ \otimes \{2^2\} =$$

$\{2^2\}$	$+\ \{321\}$	$+\ \{3^21^2\}$	$+\ \{4\}$	$+\ 2\{42\}$	$+\ 2\{42^2\}$
$+\ \{431\}$	$+\ \{4321\}$	$+\ 3\{4^2\}$	$+\ 2\{4^22\}$	$+\ \{4^22^2\}$	$+\ \{4^3\}$
$+\ 2\{51\}$	$+\ 3\{521\}$	$+\ \{52^21\}$	$+\ 2\{53\}$	$+\ \{531^2\}$	$+\ 3\{532\}$
$+\ \{53^21\}$	$+\ 4\{541\}$	$+\ 2\{5421\}$	$+\ 3\{543\}$	$+\ \{5432\}$	$+\ \{54^21\}$
$+\ 2\{5^2\}$	$+\ 2\{5^21^2\}$	$+\ \{5^22\}$	$+\ \{5^231\}$	$+\ \{5^24\}$	$+\ \{6\}$
$+\ \{61^2\}$	$+\ 6\{62\}$	$+\ \{621^2\}$	$+\ 4\{62^2\}$	$+\ \{62^3\}$	$+\ 6\{631\}$
$+\ 3\{6321\}$	$+\ \{63^2\}$	$+\ 7\{64\}$	$+\ \{641^2\}$	$+\ 10\{642\}$	$+\ 2\{642^2\}$
$+\ 3\{6431\}$	$+\ 4\{64^2\}$	$+\ 8\{651\}$	$+\ 4\{6521\}$	$+\ 6\{653\}$	$+\ 6\{6^2\}$
$+\ 2\{6^21^2\}$	$+\ 7\{6^22\}$	$+\ 3\{71\}$	$+\ 7\{721\}$	$+\ 2\{72^21\}$	$+\ 8\{73\}$
$+\ 4\{731^2\}$	$+\ 9\{732\}$	$+\ 2\{732^2\}$	$+\ 2\{73^21\}$	$+\ 13\{741\}$	$+\ 7\{7421\}$
$+\ 10\{743\}$	$+\ 8\{75\}$	$+\ 5\{751^2\}$	$+\ 15\{752\}$	$+\ 13\{761\}$	$+\ 5\{7^2\}$
$+\ 3\{8\}$	$+\ 3\{81^2\}$	$+\ 10\{82\}$	$+\ 2\{821^2\}$	$+\ 9\{82^2\}$	$+\ \{82^3\}$
$+\ 14\{831\}$	$+\ 7\{8321\}$	$+\ 6\{83^2\}$	$+\ 19\{84\}$	$+\ 6\{841^2\}$	$+\ 23\{842\}$
$+\ 24\{851\}$	$+\ 18\{86\}$	$+\ 7\{91\}$	$+\ \{91^3\}$	$+\ 14\{921\}$	$+\ 5\{92^21\}$
$+\ 15\{93\}$	$+\ 7\{931^2\}$	$+\ 20\{932\}$	$+\ 30\{941\}$	$+\ 25\{95\}$	$+\ 4\{10\ \}$
$+\ 5\{10\ 1^2\}$	$+\ 19\{10\ 2\}$	$+\ 5\{10\ 21^2\}$	$+\ 15\{10\ 2^2\}$	$+\ 28\{10\ 31\}$	$+\ 32\{10\ 4\}$
$+\ 11\{11\ 1\}$	$+\ \{11\ 1^3\}$	$+\ 24\{11\ 21\}$	$+\ 30\{11\ 3\}$	$+\ 7\{12\ \}$	$+\ 10\{12\ 1^2\}$
$+\ 28\{12\ 2\}$	$+\ 18\{13\ 1\}$	$+\ 9\{14\ \}$			

$$M_- \otimes \{2^2\} =$$

$\{2^2\}$	$+\ \{321\}$	$+\ \{3^21^2\}$	$+\ \{41^2\}$	$+\ \{42\}$	$+\ \{42^2\}$
$+\ 2\{431\}$	$+\ \{4321\}$	$+\ \{43^2\}$	$+\ 2\{4^2\}$	$+\ \{4^22\}$	$+\ \{4^22^2\}$
$+\ \{51\}$	$+\ \{51^3\}$	$+\ 3\{521\}$	$+\ \{53\}$	$+\ 2\{531^2\}$	$+\ 3\{532\}$
$+\ 2\{53^21\}$	$+\ 4\{541\}$	$+\ \{5421\}$	$+\ 3\{543\}$	$+\ \{5432\}$	$+\ 2\{5^2\}$
$+\ 3\{5^21^2\}$	$+\ 2\{5^22\}$	$+\ 2\{5^231\}$	$+\ 2\{5^24\}$	$+\ 2\{61^2\}$	$+\ 4\{62\}$
$+\ 2\{621^2\}$	$+\ 2\{62^2\}$	$+\ 7\{631\}$	$+\ 3\{6321\}$	$+\ 3\{63^2\}$	$+\ \{63^22\}$
$+\ 5\{64\}$	$+\ 2\{641^2\}$	$+\ 8\{642\}$	$+\ \{642^2\}$	$+\ 3\{6431\}$	$+\ 2\{64^2\}$
$+\ 8\{651\}$	$+\ 4\{6521\}$	$+\ 7\{653\}$	$+\ 5\{6^2\}$	$+\ 2\{6^21^2\}$	$+\ 5\{6^22\}$
$+\ 2\{71\}$	$+\ \{71^3\}$	$+\ 7\{721\}$	$+\ \{72^21\}$	$+\ 6\{73\}$	$+\ 6\{731^2\}$
$+\ 8\{732\}$	$+\ \{732^2\}$	$+\ 4\{73^21\}$	$+\ 13\{741\}$	$+\ 6\{7421\}$	$+\ 10\{743\}$
$+\ 7\{75\}$	$+\ 7\{751^2\}$	$+\ 15\{752\}$	$+\ 13\{761\}$	$+\ 5\{7^2\}$	$+\ \{8\}$
$+\ 5\{81^2\}$	$+\ 7\{82\}$	$+\ 3\{821^2\}$	$+\ 6\{82^2\}$	$+\ 16\{831\}$	$+\ 7\{8321\}$
$+\ 9\{83^2\}$	$+\ 15\{84\}$	$+\ 8\{841^2\}$	$+\ 19\{842\}$	$+\ 25\{851\}$	$+\ 15\{86\}$
$+\ 5\{91\}$	$+\ 3\{91^3\}$	$+\ 14\{921\}$	$+\ 3\{92^21\}$	$+\ 12\{93\}$	$+\ 10\{931^2\}$
$+\ 19\{932\}$	$+\ 30\{941\}$	$+\ 22\{95\}$	$+\ \{10\ \}$	$+\ 8\{10\ 1^2\}$	$+\ 14\{10\ 2\}$
$+\ 7\{10\ 21^2\}$	$+\ 10\{10\ 2^2\}$	$+\ 31\{10\ 31\}$	$+\ 26\{10\ 4\}$	$+\ 8\{11\ 1\}$	$+\ 4\{11\ 1^3\}$
$+\ 24\{11\ 21\}$	$+\ 25\{11\ 3\}$	$+\ 3\{12\ \}$	$+\ 14\{12\ 1^2\}$	$+\ 21\{12\ 2\}$	$+\ 14\{13\ 1\}$
$+\ 4\{14\ \}$					