Combinatorial Explosions in Computing Group Properties

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Abstract Certain problems in the computation of the properties of groups become combinatorially explosive and rapidly exceed the capacity and speed of any forseeable computer developments. A number of examples are considered. The way forward is seen in the development of theories of moments and asymptopia.

■ 1. Introduction

It is all too common for people to think that the solution to many problems in physics, chemistry and applied mathematics will become possible once there are computers with greater speed and memory space. The developments of the last decade tends to create unlimited optimism as to future developments. There are however many problems that are simple to state but can confidently be expected to be beyond all possible improvements in computer technology. We illustrate a number of combinatorially explosive problems that arise in computing properties of groups and applications thereof. Throughout I follow the notations associated with Macdonald[1] and Littlewood[2].

■ 2. Counting standard Young tableaux

The irreducible representations of the symmetric group $\mathcal{S}_n$ may be labelled by ordered partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p$ of the integer n. A frame $\mathcal{F}^\lambda$ may be drawn as a series of left adjusted boxes with λ_1 boxes in the first row, λ_2 boxes in the second row with λ_p boxes in the $p-$th row. A given frame $\mathcal{F}^\lambda$ with $\lambda \vdash n$ may be given a *standard* numbering by inserting in the boxes the integers $1, 2, \ldots, n$ such that the numbers in rows *and* columns are strictly increasing. Thus for $n = 4$ there are the following standard Young tableaux:-

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$$

Notice in the above examples that the number of standard tableaux for conjugate partitions is the same.

The number of standard tableaux associated with a given frame $\mathcal{F}^\lambda$ is the *dimension* f_n^λ of an irreducible representation $\{\lambda\}$ of the symmetric group $\mathcal{S}_n$. For small values of n counting Young tableaux is feasible but rapidly the number of possibilities becomes so enormous that a computer implementation seems the only way out. However, we shall shortly see that an alternative approach exists.

■ 3. Hook lengths and dimensions for $\mathcal{S}_n$

The *hook length* of a given box in a frame $\mathcal{F}^\lambda$ is the length of the right-angled path in the frame with that box as the upper left vertex. For example, the hook length of the marked box in

$$\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\\hline * & \cdot & \cdot & \cdot \\\cline{1-4} \cdot & & \\\cline{1-3} \cdot & & \\\cline{1-3} \cdot & \\\cline{1-2} \cdot \\\cline{1-1}\end{array}$$

is 8.

The task of calculating the dimensions f_n^λ is greatly simplified by application of the following theorem:-

Theorem: *To find the dimension of the representation of $\mathcal{S}_n$ corresponding to the frame F^λ, divide $n!$ by the hook length of each box of $\mathcal{F}^\lambda$.*

which is the celebrated result of Frame, Robinson and Thrall[3].

$$f_n^\lambda = \frac{n\,!}{\prod_{(i,j)\in\lambda} h_{ij}} \tag{3.1}$$

where h_{ij} is the hook length of the box in the (i,j) position.

Thus for the partition $(5\,4\,3^2\,2\,1)$ we have the hook lengths

$$\begin{array}{|c|c|c|c|c|}
\hline
10 & 8 & 6 & 3 & 1 \\
\hline
8 & 6 & 4 & 1 \\
\cline{1-4}
6 & 4 & 2 \\
\cline{1-3}
5 & 3 & 1 \\
\cline{1-3}
3 & 1 \\
\cline{1-2}
1 \\
\cline{1-1}
\end{array}$$

and hence a dimension

$$\begin{aligned} f_{18}^{543^2 21} &= \frac{18\,!}{10\times 8^2\times 6^3\times 5\times 4^2\times 3^3\times 2} \\ &= 10720710 \end{aligned}$$

It is not suggested that you check the above result by explicit enumeration!

■ 4. The staircase representations of $\mathcal{S}_n$

Among the representations of $\mathcal{S}_n$ I want to focus on the so-called *staircase* representations associated with partitions of the form $a = (a, a-1, \ldots, 1)$ where $n = \frac{a(a+1)}{2}$. The dimension of these representations is readily seen to be

$$f_n^a = \frac{n!}{\prod_{i=1}^{a-1}(2a-2i+1)^i} \tag{3.2}$$

The dimensions of the first twelve staircase representations are given in Table 3.1. There it is seen dramatically how the dimensions rapidly become astronomical and more. At $a = 14$ the staircase representation in $\mathcal{S}_{105}$ has reached

$$513782568580731957367019767803085320396632776099975918380$$

$$8656854124180549926912OO$$

$$\sim 5 \times 10^{80}$$

Suppose the Robinson, Frame, Thrall result remained undiscovered how long would it take to count the tableaux for the staircase representation of $\mathcal{S}_{105}$? Assume we have a hypersuper computer and a remarkable algorithm that counts standard tableaux at $10^9/s$. The time taken to complete the task will be $\sim 10^{71}s$. How long is that? The age of the universe is $\sim 10^{17}s$ and hence the total time taken is $\sim 10^{54}$ times the age of the universe!.

Table 4.1 Dimensions of the first twelve staircase representations.

a	n	*Dimension*
1	1	1
2	3	2
3	6	16
4	10	768
5	15	292864
6	21	1100742656
7	28	48608795688960
8	36	29258366996258488320
9	45	273035280663535522487992320
10	55	44261486084874072183645699204710400
11	66	138018895500079485095943559213817088756940800
12	78	9079590132732747656880081324531330222983622187548672000

Of course executing the very simple counting algorithm

$$t := 0; \quad \text{for } i := 1 \text{ to } N \quad \text{do } t := t + 1;$$

should rapidly convince you that counting must be avoided at all costs. The above example of determining the dimension of staircase representations of $\mathcal{S}_n$ shows clearly the importance of thinking about a problem before computing - a solution was found for a seemingly intractable problem. However, not all problems in computing properties of groups have solutions of that type. To a large extent it depends on the questions we are asking. Our example does however illustrate the way how quite simple problems can become combinatorially explosive.

Let us now turn to problems that are combinatorially explosive and by their inherent nature cannot be solved in any time small compared to the age of the universe.

■ 5. Representations of $\mathcal{S}_n$

The calculation of representation matrices pose especially severe problems even though there often exist very simple prescriptions for the evaluation of individual matrix elements. Thus for the symmetric group $\mathcal{S}_n$ Young has given a complete method for any transposition and since any element of $\mathcal{S}_n$ can be represented as a product of transpositions all representation matrices can be systematically determined by suitable matrix multiplication. The fundamental problem is that the dimensions of the representation matrices are combinatorially explosive and there is neither the time, nor the matter in the universe, to complete the calculation.

A much simpler task is the calculation of the diagonal elements of the representation matrices but that is still a combinatorially explosive problem. Evaluating the diagonal elements for the staircase representation for $\mathcal{S}_{105}$ for a *single transposition* is certainly worse than counting to 10^{80}. Even at $n = 15$ the construction of a complete representation for a single transposition of the staircase representation involves matrices of dimension 292864 which are probably barely achievable with current computers. I believe it is futile to attempt constructions of representations for large n.

■ 6. Characters of $\mathcal{S}_n$

The calculation of the characters of representations of the symmetric group proceeds much more rapidly and economically than explicit constructions of representations. This is largely due to the existence of algorithms based upon results such as the Murnaghan-Nakayama formula(cf. [4]) which avoid the need for explicit construction of any representation matrix elements. The programme **MAPLEV** will produce the complete character tables for $\mathcal{S}_n$ for $n = 6, \ldots, 13$ with the approximate timings given in Table 6.1 for a standard 80486PC. Clearly different machines will give different times but it is apparent and not surprising that the calculation of characters proceeds more rapidly than for representation matrices or explicit evaluation of traces.

Table 6.1 Time in seconds for calculating a complete character table for $\mathcal{S}_n$ using MAPLEV.

n	*time in seconds*
6	1.3
7	2.9
8	6.4
9	12.8
10	27
11	53
12	112
13	222

The time taken for the calculation of the complete set of characteristics associated with the staircase representations of $\mathcal{S}_n$ using **MAPLEV** for $n = 10, \ldots, 28$ is given in Table 6.2.

Table 6.2 Time in seconds for calculating the characters for staircase representations of $\mathcal{S}_n$

n	*time in seconds*
10	2.2
15	15
21	101
28	815

The number of elements and hence the number of matrices associated with a given representation of $\mathcal{S}_n$ increases as $n!$ whereas the number of conjugacy classes and hence number of characteristics associated with a given character increases as the number of partitions $\mathcal{P}(n)$. While $\mathcal{P}(n)$ becomes combinatorially explosive it increases more slowly than $n!$.

The above simple results and conclusions have important ramifications for applications of group theory to physical problems. In developing theories for computing properties of physical systems it is desirable, wherever possible, to express the theories in terms of characters rather than representations. This is well seen in recent emphasis on developing models of complex systems by methods of moments which make use of traces of matrices and hence characters[5,6].

■ 7. Plethysms and normal forms for tensor polynomials

The operation of S–function plethysm (or outer plethysm) was introduced by Littlewood[2] and corresponds to the formation of symmetrised powers of S–functions. It also bears a close relationship to branching rules for restriction of an irreducible representation of $\mathcal{G}l_n$ to a subgroup $\mathcal{G}l_m$[7]. Mathematicians[1] these days view the operation of plethysm as a substitution of a symmetric function into a symmetric function. Fulling *et al.*[8] have recently made extensive application of the operation of plethysm to the problem of the enumeration of the scalars formed from the Riemann tensor (of a torsionless, metric-compatible connection) by covariant differentiation, multiplication and contraction. They determined the number of independent homogeneous scalar monomials of each order and degree up to order twelve in derivatives of the metric. Wybourne and Meller[9] have extended those results to order fourteen.

The master object for enumerating the Riemann scalars is

$$\mathcal{G} \equiv \sum_{m=1}^{\infty} (t^2\{2^2\} + t^3\{32\} + t^4\{42\} + \ldots)^{\otimes m} \tag{7.1}$$

For order t^k there is a scalar for every partition involving only even parts. An equivalent problem arises in the enumeration of Weyl scalars. There one is concerned with representations of the orthogonal groups O_n. In that case the master object differs from that of Eq. (7.1) only by the replacement of the $\mathcal{G}l_n$ irreducible representations by those of O_n and a Weyl scalar occurs for every O_n scalar irreducible representation [0].

The number of Riemann and Weyl scalars for each even order of t^k (the odd orders cannot yield scalars) up to $k = 14$ is shown in Table 7.1.

Table 7.1 Numbers of Riemann and Weyl scalars up to order 14

Order	*Riemann*	*Weyl*
2	1	–
4	4	1
6	17	3
8	92	12
10	668	67
12	6721	588
14	89137	7347

Table 7.1 gives a further example of a combinatorially explosive situation and serves to focus on the need to develop alternative approaches. Had we attempted to extend the count of Weyl scalars to the sixteenth order we would need to enumerate over 250×10^6 irreducible representations of O_n. Here we might recall Major Percy MacMahon's enumeration of the number $\mathcal{P}(n)$ of ordered partitions of integers. He stopped at $n = 200$. Hardy and Ramanujan were able to develop an asymptotic form that for large n gave an exact result. Some encouragement to the development of asymptotic forms comes from noting that for sufficiently high order both the Riemann and Weyl scalars exhibit a unimodal distribution with respect to their minimal supporting dimensions. In much the same way one finds the coefficients of the terms in the SO_3 plethysms $[m]\otimes\{n\}$ approximate a Wigner type distribution as m and n become large.

■ 8. Other aspects of plethysms

Plethysms arise in many other practical problems involving not only compact groups with finite dimensional representations but also non-compact groups having infinite dimensional unitary representations. For example in the nuclear symplectic $Sp(6,R)$ shell model one is interested in symmetrised products of the fundamental representation[10]. Here the problem is not unlike that for the master object associated with the Riemann and Weyl scalars. Here the basic problem is to obtain the $Sp(6,R)$ irreducible representations that arise from the symmetrised products of A copies of the fundamental irreducible representation of $Sp(6,R)$. Since the non-trivial unitary irreducible representations of the harmonic series of $Sp(6,R)$ are necessarily of infinite dimension the number of irreducible representations contained in a symmetrised product is itself infinite. In that case progress is only possible by restriction to some finite cutoff.

The resolution of plethysms for $Sp(6,R)$ has been accomplished by exploiting a complementarity that exists between $Sp(6,R)$ and the full orthogonal group $O(A)$ together with the use of an $A-$independent reduced notation for establishing $O(A) \downarrow S(A)$ decompositions. The concept of reduced notation was introduced by Murnaghan[11] and later used by Littlewood[12] for the calculation of inner plethysms and Kronecker products for the symmetric group. A concise treatment using the properties of $S-$functions has been given by Salam and Wybourne[13].

■ Concluding remarks

I have tried to outline some of the problems associated with the computation of the properties of groups. Attention has been focussed on the well understood case of the symmetric group but similar problems arise in more general group stuctures. Most of these problems are combinatorially explosive and we cannot expect improvements in computers to have a significant impact. What is needed are new approaches to the asymptotic properties of these problems. Here the work of the St. Petersburg group of Vershik and Kerov[14]-[17] on asymptotic theory of characters of the symmetric group is perhaps pointing the way to real progress. Perhaps it is not utopia we should strive for but rather asymptopia.

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