On algebraic approaches to the genetic code

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The role of Lie groups and algebras in symmetry based models of the genetic code is considered. The two schemes, based upon the symplectic group $Sp(6)$ and the exceptional group $G(2)$ are shown to correspond to different embeddings in the group $SO(14)$. Some possible alternative schemes are sketched. Problems with considering codons being represented as fermionic or bosonic are noted. A complete listing is given of all 64-dimensional irreducible representations that can arise in the symmetric and alternating groups.

I. INTRODUCTION

Messenger ribonucleic acid (mRNA) is comprised of a sugar spine along which are attached 20 different types of amino acids. Each of these is constructed utilizing four bases (nucleotides) taken three at a time in all permutations, thereby forming *codon* (tri-nucleotide) sequences (including three sequences called 'stop', or terminating, codons). Interaction is between either two or three valence electrons. Each nucleotide is paired in DNA with another (anti-nucleotide) while each codon has a unique anti-codon. A known property of DNA and RNA is that the triplet sequences are precise and the codon assignments to amino acids, forming multiplets, are known. What is not well understood is the basis for this genetic code.

The possibility of underlying continuous symmetries being involved in the genetic code has been considered by Hornos and Hornos [1], hitherto referred to as HH. In particular they have searched for approximate symmetries in the genetic code in terms of Lie algebraic models. Specifically, they have investigated those simple Lie algebras that possess at least one irreducible representation of dimension 64, the number 64 corresponding to the $4 \times 4 \times 4$ possible codons, each involving four bases arranged in triplets, to code the 20 amino acids. HH discussed in detail group-subgroup schemes based upon the symplectic group $Sp(6)$ to describe the symmetry breaking (we abuse notation by using groups and algebras indiscriminately). More recently, Forger, Hornos and Hornos [2] (FHH) have extended their study and included an additional symmetry breaking model based upon the exceptional group $G(2)$. Herein, we comment upon certain aspects of their work, show that the two schemes correspond to two different embeddings in the group $SO(14)$ and outline some possible alternative schemes. We then comment on FHH's introduction of fermions and bosons in relation to O'Raifeartaigh's no-go theorem. Finally we give a complete listing of all 64-dimensional irreducible representations that can arise in the symmetric and alternating groups.

II. THE LIE ALGEBRAS AND LIE GROUPS

The complete list of posssible Lie algebras and their associated 64 dimensional irreducible representations is given in Table I (we exclude the trivial cases where the vector irreducible representation is of degree 64). In labelling irreducible representations we follow the natural partition labelling of King and Al-Qubanchi [3]. The first column gives the simple Lie group followed by its rank, the third column gives the natural partition labelling for the relevant 64-dimensional irreducible representation . The fourth column gives the vector irreducible representation of the group followed by its dimension while the final two columns give adjoint irreducible representation followed by its dimension. The latter number corresponds to the number of generators of the group. We note the local isomorphisms $SO(6) \sim SU(4)$, $Sp(4) \sim SO(5)$ and $SU(2) \sim Sp(2) \sim SO(3)$. Strictly speaking $SU(2)$ is the covering group of $SO(3)$, an important distinction when one is considering spin irreducible representations .

We note that the group $SO(14)$ possesses two inequivalent 64-dimensional spin irreducible representations Δ_+ and Δ_- which are conjugate to one another under an involutary outer automorphism [4]. Table I exhausts the possibilities for simple Lie groups. Other possibilities are considered towards the end of this paper.

III. $SO(14)$ AS A UNIFYING GROUP

The group $SO(14)$ has a rich set of group-subgroup structures two of which lead to a unification of the two models of HH and FHH. First we note that whereas the irreducible representations of $G(2)$ are all necessarily orthogonal, and the vector $[1]$ and adjoint $[1^2]$ irreducible representations of $SO(14)$ are also orthogonal, the spin irreducible representations $\Delta_\pm$ are complex. In the case of the symplectic group $Sp(6)$ the vector $\langle 1\rangle$ and 64-dimensional $\langle 21\rangle$ irreducible representations are symplectic and the adjoint $\langle 2\rangle$ irreducible representation is orthogonal [5,6]. These properties allow the groups $Sp(6)$ and $G(2)$ to be maximally embedded in $SO(14)$ as indeed noted by Dynkin [7]. The relevant branching rules for $SO(14) \rightarrow Sp(6)$ are:

$$\begin{aligned} &SO(14) \rightarrow Sp(6) \\ &[1] \rightarrow \langle 1^2\rangle && \text{(1a)} \\ &[1^2] \rightarrow \langle 2\rangle + \langle 21^2\rangle && \text{(1b)} \\ &\Delta_\pm \rightarrow \langle 21\rangle && \text{(1c)} \end{aligned}$$

and for $SO(14) \rightarrow G(2)$:

$$\begin{aligned} &SO(14) \rightarrow G(2) \\ &[1] \rightarrow (21) && \text{(2a)} \\ &[1^2] \rightarrow (21) + (3) && \text{(2b)} \\ &\Delta_\pm \rightarrow (31) && \text{(2c)} \end{aligned}$$

In both cases the vector $[1]$ and spin $\Delta_\pm$ irreducible representations of $SO(14)$ remain irreducible under restriction to the maximal subgroup, $Sp(6)$ or $G(2)$ while the decomposition of the adjoint irreducible representation $[1^2]$ contains the adjoint irreducible representation of the corresponding subgroup. It is these properties that lead to the conclusion that *the $Sp(6)$ and $G(2)$ symmetry models of HH and FHH correspond to different embeddings in the group $SO(14)$* .

The subgroup $Sp(6)$ is larger than $G(2)$ with the consequence that whereas $Sp(6) \supset U(1) \times SU(3)$ we have just $G(2) \supset SU(3)$. The two groups have a common $SU(3)$ subgroup. In the case of $G(2) \rightarrow SU(3)$ we have the decomposition

$$(31) \rightarrow \{32\} + \{31\} + \{2^2\} + 2\{21\} + \{2\} + \{1^2\} + \{1\} \tag{3}$$

whereas for $Sp(6) \rightarrow U(1) \times SU(3)$ we have (multiplying the $U(1)$ weights by 3 to avoid fractions of integers)

$$\begin{aligned} \langle 21\rangle \rightarrow\ & \{6\} \times \{21\} + \{2\} \times \{31\} + \{2\} \times \{2^2\} + \{2\} \times \{1\} + \{-2\} \times \{32\} \\ & + \{-2\} \times \{2\} + \{-2\} \times \{1^2\} + \{-6\} \times \{21\} \end{aligned} \tag{4}$$

Note that the presence of the $U(1)$ irreducible representations in (4) removes the degeneracy of the two $\{21\}$ irreducible representations appearing in (3). In other words the $Sp(6)$ structure is richer than that of $G(2)$.

The groups $Sp(6)$ and $G(2)$ admit several subgroup structures. HH have studied the structure

$$Sp(6) \supset Sp(4) \times SU(2) \supset SU(2) \times SU(2) \times SU(2) \tag{5}$$

and FHH also the structure

$$G(2) \supset SU(2) \times SU(2) \tag{6}$$

with further breakings of the $SU(2)$ subgroups. Under (6) one has the decomposition

$$\begin{aligned} (31) \rightarrow\ & \{5\} \times \{1\} + \{4\} \times \{2\} + \{4\} \times \{0\} + \{3\} \times \{1\} + \{2\} \times \{2\} + \{2\} \times \{0\} \\ & + \{1\} \times \{3\} + \{1\} \times \{1\} \end{aligned} \tag{7}$$

Comparison with (3) shows the $G(2) \rightarrow SU(3)$ and $G(2) \supset SU(2) \times SU(2)$ schemes yield the same number of representations but different degeneracies.

In general, if the vector irreducible representation $\{1\}$ decomposes under $SU(n) \rightarrow U(1) \times SU(n-1)$ as

$$\{1\} \rightarrow \{1\} \times \{1\} + \{-n+1\} \times \{0\} \tag{8}$$

then an arbitrary irreducible representation $\{\lambda\}$ of $SU(n)$ decomposes as

$$\{\lambda\} \to \sum_m \{-nm + \omega_\lambda\} \times \{\lambda/m\} \tag{9}$$

where ω_λ is the weight of the partition (λ), with $\{\lambda/m\}$ restricted to partitions into at most $(n-1)$ non-zero parts and remembering that in $SU(p)$ for an irreducible representation $\{\rho\}$ involving p non-zero parts we have the equivalence

$$\{\rho_1, \rho_2, \ldots, \rho_p\} \equiv \{\rho_1 - \rho_p, \rho_2 - \rho_p, \ldots, \rho_{p-1} - \rho_p, 0\} \tag{10}$$

Using (10) we can decompose the $SU(3)$ irreducible representations in (3) under $SU(3) \to U(1) \times SU(2)$ to get under $G(2) \to SU(3) \to U(1) \times SU(2)$

$$\begin{aligned}(31) \to\ & \{5\} \times \{1\} + \{4\} \times \{2\} + \{4\} \times \{0\} + 2\{3\} \times \{1\} + 2\{2\} \times \{2\} + 2\{2\} \times \{0\} \\ & + \{1\} \times \{3\} + 3\{1\} \times \{1\} + 2\{0\} \times \{2\} + 2\{0\} \times \{0\} + \{-1\} \times \{3\} \\ & + 3\{-1\} \times \{1\} + 2\{-2\} \times \{2\} + 2\{-2\} \times \{0\} + 2\{-3\} \times \{1\} + \{-4\} \times \{2\} \\ & + \{-4\} \times \{0\} + \{-5\} \times \{1\}\end{aligned} \tag{11}$$

which is, not surprisingly, the same $U(1) \times SU(2)$ content obtained by FHH in Step 3 of their Table III.

The group $SO(14)$ admits many different subgroup structures. Only a few have been explored in detail. The structures in (1) and (2) are unique in as much as the $SO(14)$ irreducible representation remains irreducible in the first stage of the symmetry reduction. All other structures (apart from $SO(14) \supset SO(13)$) involve a reduction into two or more irreducible representations at the first stage of the symmetry reduction. Thus, for example, one has structures such as

$$SO(14) \supset SO(7) \times SO(7) \supset G(2) \times G(2) \subset G(2) \tag{12}$$

with the vector irreducible representation of $SO(14)$ decomposing as

$$[1] \to [1] \times [0] + [0] \times [1] \to (10) \times (00) + (00) \times (10) \to 2(10) \tag{13}$$

and the spin irreducible representations as

$$\Delta_\pm \to \Delta \times \Delta \to ((10) + (00)) \times ((10) + (00)) \to (21) + (20) + 3(10) + 2(00) \tag{14}$$

The group $SO(13)$ is of less significance as the vector irreducible representation is of dimension 13 leading to rather uninteresting subgroup structures. It would appear from our preceding remarks that the Lie group of major interest remains as HH's original choice of $Sp(6)$.

IV. OTHER $SP(6)$ MODELS

The group $Sp(6)$ admits a variety of subgroup structures. Most are of little interest apart from those already considered by HH and FHH and the structure

$$Sp(6) \supset U(1) \times SU(3) \supset U(1) \times U(1) \times SU(2) \tag{15}$$

The first step of the symmetry reduction is given in (4). Each irreducible representation given in (4) decomposes in the second step according to the results given in Table II. Every one of the 30 $U(1) \times U(1) \times SU(2)$ multiplets has unique labels. There is a one-to-one correspondence between the final degeneracies of this model with that of the G_2 model of FHH. The difference is in the earlier breaking of the symmetry. As already noted there are two distinct G_2 models, namely that based upon $G_2 \to SU(2) \times SU(2)$ and used by FHH and that based upon $G_2 \to SU(3)$.

A typical state in the $Sp(6) \to U(1) \times SU(3) \to U(1) \times SU(2) \times U(1)$ may be uniquely labelled as

$$|\langle 21\rangle Y_1 \{\lambda\} Y_2 \{m\} Y_3\rangle \tag{16}$$

The Y_1 , Y_2 and Y_3 labels (to within a normalisation factor) are associated with the three $U(1)$ subgroups and are analogous to hypercharge. The third $U(1)$ comes from the $SU(2) \to U(1)$ reduction required to give a complete set of basis states. It is assumed that the states are degenerate with respect to Y_3. The $SU(3)$ irreducible representation is labelled by a two part partition $\{\lambda\}$. The irreducible representations of $SU(2)$ are labelled by $\{m\}$ and are associated with a degeneracy of $(m+1)$ with Y_3 distinguishing the degenerate states.

The multiplicities may be represented by the eigenvalues of the operator

$$\mathcal{O} = a_0 + a_1 Y_1 + a_2 C_2(SU(3)) + a_3 Y_2 + a_4 m(m+2) \tag{17}$$

The eigenvalues of $C_2(SU(3))$ may be taken, to within a normalisation factor, as [8]

$$C_2(\lambda_1, \lambda_2) = 2(\lambda_1^2 + \lambda_2^2 - \lambda_1^2\lambda_2^2 + 3\lambda_1) \tag{18}$$

The invariant operator (17) has the same number of parameters as FHH's $G(2)$ model and one less than that of the original HH $Sp(6)$ model. Nevertheless, the operator given here represents a different mode of symmetry breaking. The complete set of states is displayed in Table III.

V. OF BOSONS AND FERMIONS

The algebraic models proposed in HH and FHH reflect experience with algebraic models in nuclear and particle physics. Such ideas must be developed with caution when taken over to biological problems. The introduction of fermionic and bosonic representations in FHH goes against the usual ideas of particle physics where one has O'Raifeartaigh's no-go theorem [9] that rules out the possibility of combining fermions and bosons into a common multiplet, or irreducible representation , of a Lie group—that becomes possible only in going to super-Lie groups which introduce both commutative and anticommutative variables. Thus, in the standard $SU(3)$ model of the baryon octet, the adjoint irreducible representation $\{21\}$ of $SU(3)$ contains isospin multiplets with both integer and half integer isospin but the baryons are all fermions. Similarly in the meson nonets both integer and half integer isospin occur but the mesons are all bosons. The relevant group for isospin is $SU(2)$, the covering group of $SO(3)$. The possibility of using super algebras in analysing the genetic code has been considered by Bashford *et al.* [10].

VI. SYMMETRIC AND ALTERNATING GROUPS

Scant attention has been paid to the finite groups. Irreducible representations of dimension 64 are relatively rare among the finite groups. None of the crystallographic point groups contain irreducible representations of degree $\rangle 6$. This leaves the symmetric, $S(n)$, and alternating, $A(n)$, groups as the most likely candidates. In those cases one can give an exhaustive list. For the ordinary irreducible representations the only cases for all n are for $n = 8$ and $n = 65$, specifically

$$\begin{aligned} S(8) :- &\quad \{521\}, \quad \{321^3\} \\ A(8) :- &\quad [521] \\ S(65) :- &\quad \{641\}, \quad \{21^{63}\} \\ A(65) :- &\quad [641] \end{aligned}$$

while for the spin irreducible representations we are limited to the cases

$$\begin{aligned} S(13) :- &\quad \{\Delta\} \\ S(14) :- &\quad \{\Delta_\pm\} \\ A(14) :- &\quad [\Delta] \\ A(15) :- &\quad [\Delta_\pm] \end{aligned}$$

It is difficult to motivate a symmetry breaking based upon these groups though we note the explicit appearance of permutational invariance in the closing stages of FHH.

VII. CONCLUDING REMARKS

It would appear from the present work that the most suitable Lie group for describing symmetry breaking models of the genetic code is the symplectic group $Sp(6)$ and that at this stage in the development more than one $Sp(6)$ model is possible. The possibilities of using finite groups appear to be quite restrictive as long as it is assumed that the symmetry breaking starts with a 64-dimensional irreducible representation . As FHH rightly note *"symmetry considerations alone cannot replace a microscopic model but just establish a general background."* This parallels the corresponding difficulty that has been experienced in physics with the interacting boson model of nuclei. HH and FHH have raised some important issues. Developing a microscopic model remains the major task for future work. Some preliminary steps in that direction have been given elsewhere [11].

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TABLE I. The Lie groups possessing a 64-dimensional irreducible representation .

Group	rank	λ_{64}	V	dim V	Ad	dim Ad
$SO(14)$	7	$\Delta_{\pm}$	$[1]$	14	$[1^2]$	91
$SO(13)$	6	Δ	$[1]$	13	$[1^2]$	78
$Sp(6)$	3	$\langle 21\rangle$	$\langle 1\rangle$	6	$\langle 2\rangle$	21
$SO(6)$	3	$[21]$	$[1]$	6	$[1^2]$	15
$SU(4)$	3	$\{321\}$	$\{1\}$	4	$\{21^2\}$	15
$G(2)$	2	(31)	$(1,0)$	7	(21)	14
$SO(5)$	2	$[\Delta 21]$	$[1]$	5	$[1^2]$	10
$Sp(4)$	2	$\langle 41\rangle$	$\langle 1\rangle$	4	$\langle 2\rangle$	10
$SU(3)$	2	$\{63\}$	$\{1\}$	3	$\{21\}$	8
$Sp(2)$	1	$\langle 63\ \rangle$	$\langle 1\rangle$	2	$\langle 2\rangle$	3
$SU(2)$	1	$\{63\ \}$	$\{1\}$	2	$\{2\}$	3
$SO(3)$	1	$[\Delta 31\ ]$	$[1]$	3	$[1]$	3

TABLE II. $U(1) \times SU(3) \rightarrow U(1) \times U(1) \times$ branching rules[a].

(8) $\{6\} \times \{21\} \rightarrow$	$\{6\} \times \{3\} \times \{1\}$	$+ \{6\} \times \{0\} \times \{2\}$	$+ \{6\} \times \{0\} \times \{0\}$
	$+ \{6\} \times \{-3\} \times \{1\}$		
(8) $\{-6\} \times \{21\} \rightarrow$	$\{-6\} \times \{3\} \times \{1\}$	$+ \{-6\} \times \{0\} \times \{2\}$	$+ \{-6\} \times \{0\} \times \{0\}$
	$+ \{-6\} \times \{-3\} \times \{1\}$		
(15) $\{2\} \times \{31\} \rightarrow$	$\{2\} \times \{4\} \times \{2\}$	$+ \{2\} \times \{1\} \times \{3\}$	$+ \{2\} \times \{1\} \times \{1\}$
	$+ \{2\} \times \{-2\} \times \{2\}$	$+ \{2\} \times \{-2\} \times \{0\}$	$+ \{2\} \times \{-5\} \times \{1\}$
(15) $\{-2\} \times \{32\} \rightarrow$	$\{-2\} \times \{5\} \times \{1\}$	$+ \{-2\} \times \{2\} \times \{2\}$	$+ \{-2\} \times \{2\} \times \{0\}$
	$+ \{-2\} \times \{-1\} \times \{3\}$	$+ \{-2\} \times \{-1\} \times \{1\}$	$+ \{-2\} \times \{-4\} \times \{2\}$
(6) $\{2\} \times \{2^2\} \rightarrow$	$\{2\} \times \{4\} \times \{0\}$	$+ \{2\} \times \{1\} \times \{1\}$	$+ \{2\} \times \{-2\} \times \{2\}$
(6) $\{-2\} \times \{2\} \rightarrow$	$\{-2\} \times \{2\} \times \{2\}$	$+ \{-2\} \times \{-1\} \times \{1\}$	$+ \{-2\} \times \{-4\} \times \{0\}$
(3) $\{2\} \times \{1\} \rightarrow$	$\{2\} \times \{1\} \times \{1\}$	$+ \{2\} \times \{-2\} \times \{0\}$	
(3) $\{-2\} \times \{1^2\} \rightarrow$	$\{-2\} \times \{2\} \times \{0\}$	$+ \{-2\} \times \{-1\} \times \{1\}$	

[a]The dimensions of the $U(1) \times SU(3)$ irreducible representations are placed in brackets () in the leftmost column. The dimensions of the $SU(2)$ irreducible representations $\{m\}$ are just $(m+1)$.

TABLE III. The complete set of states in the $Sp(6) \rightarrow U(1) \times U(1) \times SU(2)$ scheme.

$\|\langle 21\rangle \pm 6\{21\} \pm 3\{1\} \pm 1\rangle$	$\|\langle 21\rangle \pm 6\{21\}0\{2\} \pm 2\rangle$
$\|\langle 21\rangle \pm 6\{21\}0\{2\}0\rangle$	$\|\langle 21\rangle \pm 6\{21\}0\{0\}0\rangle$
$\|\langle 21\rangle 2\{31\}4\{2\} \pm 2\rangle$	$\|\langle 21\rangle 2\{31\}4\{2\}0\rangle$
$\|\langle 21\rangle 2\{31\}1\{3\} \pm 3\rangle$	$\|\langle 21\rangle 2\{31\}1\{3\} \pm 1\rangle$
$\|\langle 21\rangle 2\{31\}1\{1\} \pm 1\rangle$	$\|\langle 21\rangle 2\{31\} - 2\{2\} \pm 2\rangle$
$\|\langle 21\rangle 2\{31\} - 2\{2\}0\rangle$	$\|\langle 21\rangle 2\{31\} - 2\{0\}0\rangle$
$\|\langle 21\rangle 2\{31\} - 5\{1\} \pm 1\rangle$	
$\|\langle 21\rangle - 2\{32\}5\{1\} \pm 1\rangle$	$\|\langle 21\rangle - 2\{32\}2\{2\} \pm 2\rangle$
$\|\langle 21\rangle - 2\{32\}2\{2\}0\rangle$	$\|\langle 21\rangle - 2\{32\}2\{0\}0\rangle$
$\|\langle 21\rangle - 2\{32\} - 1\{3\} \pm 3\rangle$	$\|\langle 21\rangle - 2\{32\} - 1\{3\} \pm 1\rangle$
$\|\langle 21\rangle - 2\{32\} - 1\{1\} \pm 1\rangle$	$\|\langle 21\rangle - 2\{32\} - 4\{2\} \pm 2\rangle$
$\|\langle 21\rangle - 2\{32\} - 4\{2\}0\rangle$	
$\|\langle 21\rangle 2\{2^2\}4\{0\}0\rangle$	$\|\langle 21\rangle 2\{2^2\}1\{1\} \pm 1\rangle$
$\|\langle 21\rangle 2\{2^2\} - 2\{2\} \pm 2\rangle$	$\|\langle 21\rangle 2\{2^2\} - 2\{2\}0\rangle$
$\|\langle 21\rangle - 2\{2\}2\{2\} \pm 2\rangle$	$\|\langle 21\rangle - 2\{2\}2\{2\}0\rangle$
$\|\langle 21\rangle - 2\{2\} - 1\{1\} \pm 1\rangle$	$\|\langle 21\rangle - 2\{2\} - 4\{0\}0\rangle$
$\|\langle 21\rangle 2\{1\}1\{1\} \pm 1\rangle$	$\|\langle 21\rangle 2\{1\} - 2\{0\}0\rangle$
$\|\langle 21\rangle - 2\{1^2\} - 1\{1\} \pm 1\rangle$	$\|\langle 21\rangle - 2\{1^2\}2\{0\}0\rangle$