Bosons and fermions in a one-dimensional harmonic oscillator potential

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Abstract. The symmetry properties of bosons and fermions in a one-dimensional harmonic oscillator are explored. It is shown that there is a one-to-one correspondence between the counts of the maximal spin states states formed by $N-$non-interacting bosons and fermions in a one-dimensional harmonic oscillator.

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1. Introduction

It is sometimes said that there are only two solvable problems in physics, the harmonic oscillator and the one centre Kepler problem, and in reality even they are approximations. Nevertheless, the harmonic oscillator, both in its classical and quantum formulations continues to fascinate and, at times, to give new insights into old physics problems. As in most areas of physics symmetry concepts play a key role. Even the classical one-dimensional harmonic oscillator has a surprisingly large symmetry group[1], $SL(4,\Re)$. The $n-$dimensional isotropic harmonic oscillator has the metaplectic group, $Mp(2n)$ as its dynamical group[2] which is the double covering group of the non-compact group $Sp(2n,\Re)$. These groups are characterised by non-trivial infinite dimensional unitary representations. The complete set of states for a single particle, boson or fermion, in a $n-$dimensional isotropic harmonic oscillator span the single irreducible representation $\tilde{\Delta}$ of $Mp(2n)$. Under the restriction $Mp(2n) \to Sp(2n,\Re)$ the irreducible representation $\tilde{\Delta}$ splits into the sum of two infinite dimensional irreducible representations $\Delta_{\pm}$ of $Sp(2n,\Re)$ viz.

$$\tilde{\Delta} \to \Delta_+ + \Delta_- \tag{1}$$

In general, we shall label the so-called harmonic series irreducible representations of $Sp(2n,\Re)$ with the notation[2] $\langle \frac{1}{2}k(\lambda)\rangle$ $(\lambda) = (\lambda_1, \lambda_2, \ldots)$ for which the conjugate partition[3] $(\tilde{\lambda}) = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ satisfies the constraints[2]

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq k \qquad \text{and} \qquad \tilde{\lambda}_1 \leq n \tag{2}$$

In such a notation

$$\Delta_+ \equiv \langle \tfrac{1}{2}(0)\rangle \qquad \text{and} \qquad \Delta_- \equiv \langle \tfrac{1}{2}(1)\rangle \tag{3}$$

The group $Sp(2n,\Re)$ has as its maximal compact subgroup the unitary group $U(n)$ which is known as the *degeneracy group* of the $n-$dimensional harmonic oscillator. Under the restriction $Sp(2n,R) \to U(n)$ one has the decompositions

$$\Delta_+ = \langle \tfrac{1}{2}(0)\rangle \to \varepsilon^{\frac{1}{2}}(\{0\} + \{2\} + \{4\} + \ldots) \tag{4a}$$

$$\Delta_- = \langle \tfrac{1}{2}(1)\rangle \to \varepsilon^{\frac{1}{2}}(\{1\} + \{3\} + \{5\} + \ldots) \tag{4b}$$

Thus Δ_+ covers the infinite series of *even* parity states and Δ_- the infinite series of *odd* parity states for a single particle in a $n-$dimensional harmonic oscillator. Noting (1) and (4) we can succinctly write the decomposition for the $\tilde{\Delta}$ irreducible representation of $Mp(2n)$ under $Mp(2n) \to U(n)$ as[2]

$$\tilde{\Delta} \to \varepsilon^{\frac{1}{2}} M \tag{5}$$

with

$$M = \sum_{m=0}^{\infty} \{m\} \tag{6}$$

With the above notation established we can now turn to the major purpose of this note, the special case of bosons and fermions in a one-dimensional harmonic oscillator.

2. Counting states for N-noninteracting bosons and fermions

For a one-dimensional harmonic oscillator the degeneracy group is $U(1)$, but all the irreducible representations of $U(1)$ are one-dimensional. For $N-$non-interacting particles in a one-dimensional harmonic oscillator we wish to count the number of *symmetric* states for bosons and *antisymmetric* states for fermions. At the $U(1)$ level this is equivalent to evaluating the terms in the expansion of the respective plethysms[4]

$$M \otimes \{N\} = \sum_{k=0}^{\infty} g_N^k \{k\} \qquad \text{bosons} \tag{7b}$$

$$M \otimes \{1^N\} = \sum_{\ell=\frac{N(N-1)}{2}}^{\infty} c_N^\ell \{\ell\} \qquad \text{fermions} \tag{7f}$$

Due to the simple nature of the group $U(1)$ the expansion coefficients g_N^k and c_N^ℓ may be completely determined in terms of the enumeration of partitions with g_N^k being the number of partitions of the integer k into at most N parts with repetitions and null parts allowed while c_N^ℓ is the number of partitions of ℓ into N distinct parts, including the null part.

We can map of one of the sets of partitions into the other by adding, or subtracting, $\rho_N = (N-1,\ldots,2,1,0)$. Adding ρ_N to the partitions of k into at most N parts, converts them into partitions, all of whose parts are distinct. Hence

$$c_N^\ell = g_N^k \qquad \text{if} \quad \ell = k + \tfrac{N(N-1)}{2} \tag{8}$$

For example

$$M \otimes \{4\} \supset \{0\} + \{1\} + 2\{2\} + 3\{3\} + 5\{4\} + 6\{5\} + 9\{6\} + 11\{7\} + \ldots \tag{9a}$$

$$M \otimes \{1^4\} \supset \{6\} + \{7\} + 2\{8\} + 3\{9\} + 5\{10\} + 6\{11\} + 9\{12\} + 11\{13\} + \ldots \tag{9b}$$

noting that $c_4^{k+6} = g_4^6$. For g_4^7 and c_4^{13} we have the respective sets of 11 partitions

$$g_4^7 \qquad (2^31) + (321^2) + (32^2) + (3^21) + (41^3) + (421) + (43) + (51^2) + (52) + (61) + (7) \tag{10a}$$

$$c_4^{13} \qquad (5431) + (6421) + (643) + (652) + (7321) + (742) + (751) + (832) + (841) + (931) + (10\ 21) \tag{10b}$$

adding $(3,2,1,0)$ to each partition in (10a) gives the partitions in (10b).

Thus we can conclude that there is a one-to-one correspondence between the counts of the states formed by $N-$non-interacting bosons and fermions in a one-dimensional harmonic oscillator. This has the known consequence[5,6] that in such a situation the thermodynamic properties of $N-$non-interacting bosons and fermions are equivalent apart from a shift in the groundstate energy.

3. The $Sp(2,\Re)$ symmetry

As already mentioned the infinite set of states of the one-dimensional harmonic oscillator span the reducible representation $\tilde{\Delta} = \Delta_+ + \Delta_-$ of $Sp(2,\Re)$. Suppose

$$\tilde{\Delta} \otimes \{N\} = \sum_{k=0}^{\infty} s_N^k \langle \tfrac{N}{2}(k)\rangle \tag{11a}$$

$$\tilde{\Delta} \otimes \{1^N\} = \sum_{\ell=\frac{N(N-1)}{2}}^{\infty} a_N^\ell \langle \tfrac{N}{2}(\ell)\rangle \tag{11b}$$

It follows from the previous section and the known decomposition rules for $Sp(2,\Re) \rightarrow U(1)$ that[2]

$$s_N^k = g_N^k - g_N^{k-2} \tag{12a}$$

$$a_N^\ell = g_N^{\ell-\frac{N(N-1)}{2}} - g_N^{\ell-\frac{N(N-1)}{2}-2} \tag{12b}$$

Thus the $Sp(2,\Re)$ content of the two plethysms differ simply by

$$\ell \rightarrow \ell - \tfrac{N(N-1)}{2} \tag{13}$$

i.e.

$$a_N^\ell = s_N^{\ell-\frac{N(N-1)}{2}} \tag{14}$$

which again reflects the boson-fermion symmetry of the one-dimensional harmonic oscillator.

As an example, for $N = 5$ we have

$$\begin{aligned}\tilde{\Delta} \otimes \{5\} = \; & \langle s2(0)\rangle && + \langle s2(1)\rangle && + \langle s2(2)\rangle && + 2\langle s2(3)\rangle \\ & + 3\langle s2(4)\rangle && + 4\langle s2(5)\rangle && + 5\langle s2(6)\rangle && + 6\langle s2(7)\rangle \\ & + 8\langle s2(8)\rangle && + 10\langle s2(9)\rangle && + 12\langle s2(10)\rangle && + \ldots\end{aligned} \tag{15a}$$

and

$$\begin{aligned}\tilde{\Delta} \otimes \{1^5\} = \; & < s2(10) > && + < s2(11) > && + < s2(12) > && + 2 < s2(13) > \\ & + 3 < s2(14) > && + 4 < s2(15) > && + 5 < s2(16) > && + 6 < s2(17) > \\ & + \ldots\end{aligned} \tag{15b}$$

hence we see, for example, that

$$a_5^{16} = s_5^6 = 5$$

4. Inclusion of spin

In the preceding we have assumed in each case the bosons or fermions have been prepared in states involving a single spin component. In some experimental situations such a preparation is possible. In general the full spin needs to be taken into account by considering direct products of the spin group $SU(2)$ with the groups appropriate to the description of the one-dimensional space. Such an extension is relatively simple. One then finds that there is a restricted boson-fermion correspondence, namely that between boson and fermion states of maximal spin multiplicity.

5. Concluding remarks

We have shown that there is a is a qualified one-to-one correspondence between the counts of the states formed by $N-$non-interacting bosons and fermions in a one-dimensional harmonic oscillator. This is consistent with the known thermodynamic properties of such systems for suitably prepared systems..

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