Dyskretne funkcje Wignera i bazy komplementarne

Wigner 1932

$$\psi \in \mathcal{H} = L^2(\mathbb{R})$$

$$\psi = \psi(x)$$

$$\psi(x) \longrightarrow W(x,p)$$

$$W(x,p) = \frac{1}{\pi\hbar}\int dy\, \bar{\psi}(x+y)\,\psi(x-y)\, e^{2ipy/\hbar}$$

1) $W(x,p) \in \mathbb{R}$ $\quad(\psi \in \mathbb{C}\,!)$

2) $\int dx\,dp\, W(x,p) = 1$ $\quad\left(\int |\psi|^2 dx = 1\right)$

3) Uwaga! $\quad W(x,p) \not\geq 0$

$$|W(x,p)| \leq \frac{2}{h}$$

Tw. (Hudson 1974) ODWR!

$$W(x,p) \geq 0 \iff \psi \text{ - f. Gaussa.}$$

$$W(x,p) = \frac{1}{\pi\hbar} \int \overline{\psi}(x+y)\, \psi(x-y)\, e^{2ipy/\hbar}\, dy$$

$$= \frac{1}{\pi\hbar} \int \langle x-y \underbrace{|\psi\rangle\langle\psi|}\, x+y\rangle\, e^{2ipy/\hbar}\, dy$$

$\psi(x) \equiv \langle x|\psi\rangle$

$$|\psi\rangle\langle\psi| \longrightarrow \hat{\rho}$$

$$\boxed{\hat{\rho} \longrightarrow W(x,p) = \frac{1}{\pi\hbar} \int \langle x-y|\hat{\rho}|x+y\rangle\, e^{2ipy/\hbar}}$$

— * —

$W(x,p)$ <u>nie jest</u> rozkładem prawdop.

ale

$$\left.\begin{aligned} \int dp\, W(x,p) &= |\psi(x)|^2 \\ \int dx\, W(x,p) &= |\tilde{\varphi}(p)|^2 \end{aligned}\right\} \text{rozkłady brzegowe}$$

$$\psi \longrightarrow \hat{\rho}$$

$$\int dp \;\ldots\; = \langle x|\hat{\rho}|x\rangle$$

$$\int dx \;\ldots\; = \langle p|\hat{\rho}|p\rangle$$

$$\psi_1 \longrightarrow W_1$$

$$\psi_2 \longrightarrow W_2$$

$$\int W_1 \cdot W_2 \, d\Gamma = \frac{1}{h} |\langle \psi_1 | \psi_2 \rangle|^2$$

prawd. przejścia.

Inne quasi-rozkłady ważne dla optyki kwantowej

$|\alpha\rangle$ - stan koherentny

$$\hat{\rho} \longrightarrow \underline{\underline{Q(\alpha)}} = \S \langle \alpha | \hat{\rho} | \alpha \rangle$$

$$\hat{\rho} \longrightarrow \S \, \rho = \int \underline{\underline{P(\alpha)}} \, |\alpha\rangle\langle\alpha| \, d^2\alpha$$

Q & P - reprezentacje !

Glauber & Sudershan

$$W(x,p) = \frac{1}{\pi\hbar}\int \langle x-y|\hat{\varrho}|x+y\rangle \, e^{2ipy/\hbar}\, dy$$

$$= \cancel{\frac{1}{\pi\hbar}}\,\cancel{2\pi}\,\mathrm{Tr}\left[\hat{\varrho}\,\frac{1}{\pi\hbar}\int |x+y\rangle\langle x-y|\, e^{2ipy/\hbar}\, dy\right]$$

$$W(x,p) = \frac{1}{2\pi\hbar}\,\mathrm{Tr}\left[\hat{\varrho}\,\hat{A}(x,p)\right]$$

$$\hat{A}(x,p) := 2\int |x+y\rangle\langle x-y|\, e^{2ipy/\hbar}\, dy$$

$$\Gamma \qquad \vartheta(x,p) \longrightarrow \hat{A}(x,p)$$

— * —

$$\boxed{\hat{A}(0,0) = 2\int |y\rangle\langle -y|\, dy}$$ operator parzystości

$$\hat{A}(0,0)|x\rangle = 2\int |-y\rangle\underbrace{\langle y|x\rangle}_{\delta(x-y)}\, dy$$

$$= 2|-x\rangle .$$

$$\hat{D}(u,v) := e^{i(u\hat{x} - v\hat{p})/\hbar}$$ operator przesunięcia

$$\hat{A}(x,p) = \hat{D}(x,p)\,\hat{A}(0,0)\,\hat{D}(x,p)^{*}$$

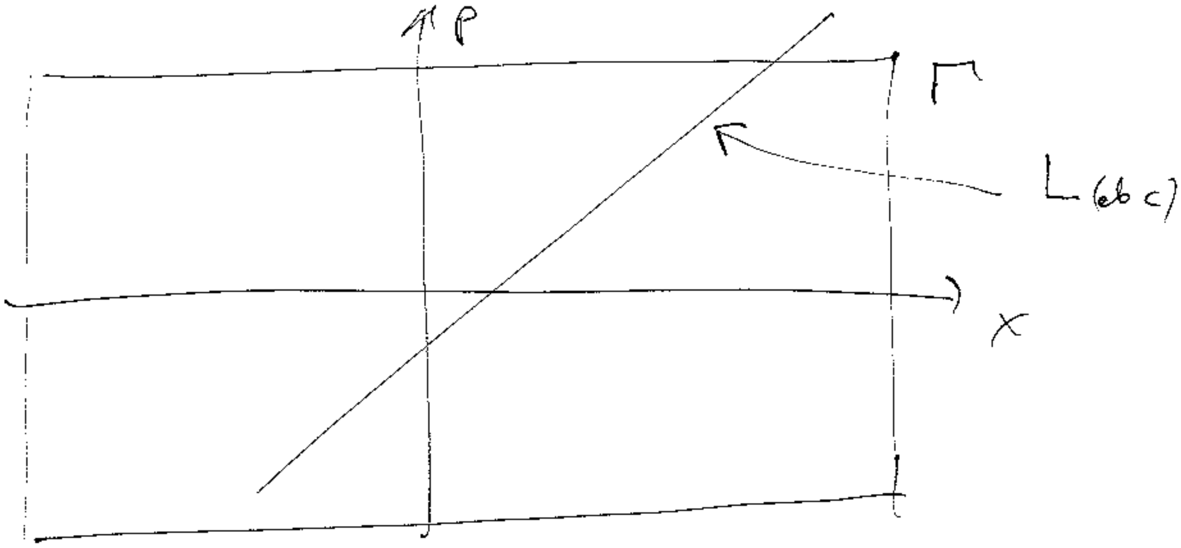
Własności:

1) $\mathrm{Tr}\,\hat{A}(x,p) = 1$

2) $\mathrm{Tr}\left[\hat{A}(x,p)\,\hat{A}(x',p')\right] = 2\pi\hbar\,\delta(x-x')\,\delta(p-p')$

$$\hat{\rho} \longrightarrow W(x,p) = \frac{1}{2\pi\hbar}\,\mathrm{Tr}\left[\hat{\rho}\,\hat{A}(x,p)\right]$$

$$W \longrightarrow \hat{\rho} = \int W(x,p)\,\hat{A}(x,p)\,dx\,dp\,.$$



$$L_{(abc)} \longrightarrow ax + bp = c$$

$$\frac{1}{2\pi\hbar}\int_{L_{(abc)}} \hat{A} = \hat{P}_{(abc)}$$ – 1-wym. projektor

$$\hat{P}_{(abc)} = |\psi_{(abc)}\rangle\langle\psi_{(abc)}|$$

$$\psi_{(abc)} = ?$$

$$L_{(abc)} \longrightarrow \hat{Q}_{(ab)} = a\hat{x} + b\hat{p}$$

$$\boxed{\hat{Q}_{(ab)}|\psi_{(abc)}\rangle = c\,|\psi_{(abc)}\rangle}$$

$$L_{(abc)} \longmapsto \hat{P}_{(abc)}$$

$$L_{(abc)} \parallel L'_{(abc')} \Longleftrightarrow \hat{P}_{(abc)} \perp \hat{P}^{\#}_{(abc')}$$

$$\int dc \, \hat{P}_{(abc)} = \hat{\mathbb{I}}$$

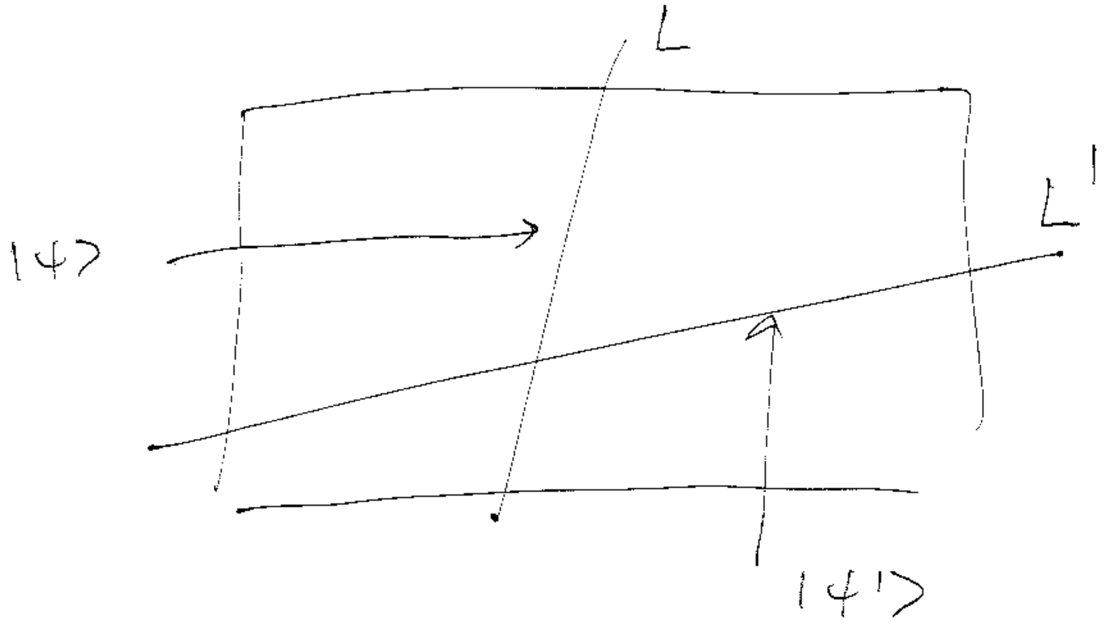
TOMOGRAFIA.

pomiar $\quad \hat{P}_{(abc)} \longrightarrow \mathrm{Tr}\left[\hat{\varrho}\, \hat{P}_{(abc)}\right] = p_{(abc)}$

$\boxed{p_{(abc)}}$ – rozkład prawd.

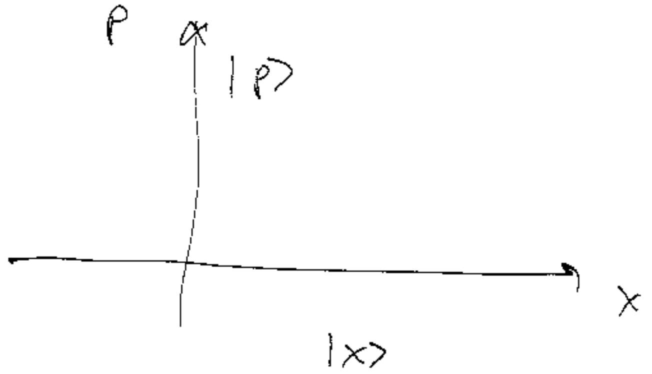
LAB

$$p_{(abc)} \xrightarrow[\text{Radona}]{\text{odwr. tr.}} \varrho_W \longleftrightarrow \hat{\varrho} \, .$$



$$|\langle\psi|\psi'\rangle| = \text{const}$$ nie zależy od L & L' !

<u>Przykład</u>



$$\langle p|x\rangle = \frac{1}{2\pi\hbar} e^{-ipx/\hbar}$$

$$\boxed{|\langle p|x\rangle| = \frac{1}{2\pi\hbar}} \quad !$$

Dyskretne Funkcje Wignera

$\dim \mathcal{H} = d \longrightarrow \mathcal{H} \cong \mathbb{C}^d$.

$\{e_0, \ldots, e_{d-1}\}$, $\{f_0, \ldots, f_{d-1}\}$ – bazy ortonormalne

$\langle e_\alpha | e_\beta \rangle = \langle f_\alpha | f_\beta \rangle = \delta_{\alpha\beta}$

<u>Def</u>. Bazy $\{e_\alpha\}$ & $\{f_\alpha\}$ nazywamy <u>komplementarnymi</u> (MUB – mutually unbiased)

$$|\langle e_\alpha | f_\beta \rangle|^2 = \frac{1}{d}.$$

<u>Fakty</u>:

1) $\#\text{MUBs} \leq d+1$.

2) $d = p^\alpha \Rightarrow \#\text{MUBs} = d+1$

3) $d \neq p^\alpha$ nie wiemy, czy istnieje $d+1$ czy też mniej.

<u>EX</u> dla $d = 6$ ~~znanych~~ znane są tylko 3 MUBs!

Beispiel: $\quad d = \cancel{3}\,2$

$\sigma_x, \sigma_y, \sigma_z$

$$\sigma_z \longrightarrow |0\rangle, |1\rangle$$

$$\left.\begin{array}{l} \sigma_z|0\rangle = |0\rangle \\ \sigma_z|1\rangle = -|1\rangle \end{array}\right]$$

$$\left.\begin{array}{lll} \sigma_z \longrightarrow & \begin{pmatrix}1\\0\end{pmatrix}, & \begin{pmatrix}0\\1\end{pmatrix} \\ \sigma_x \longrightarrow & \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, & \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \\ \sigma_y \longrightarrow & \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, & \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \end{array}\right\} \underline{\text{3 MUBs}}$$

Dlaczego MUBs są ważne ?

Przypuśćmy, że istnieje <u>max</u> # $= \underline{\underline{d+1}}$

$d+1$ – baz komplementarnych

$$\left\{ e_k^{(\alpha)} \right\} \qquad \begin{array}{l} \alpha = 0, 1, \dots, d \\ k = 0, 1, \dots, d-1 \end{array}$$

$$\hookrightarrow \quad \hat{P}_k^{(\alpha)} = | e_k^{(\alpha)} \rangle \langle e_k^{(\alpha)} |$$

$$\forall \; \hat{\rho} \longrightarrow \mathrm{Tr}\left[\hat{\rho} \, \hat{P}_k^{(\alpha)} \right] =: p_k^{(\alpha)}$$

$d+1$ – rozkładów pr. $p_k^{(\alpha)}$

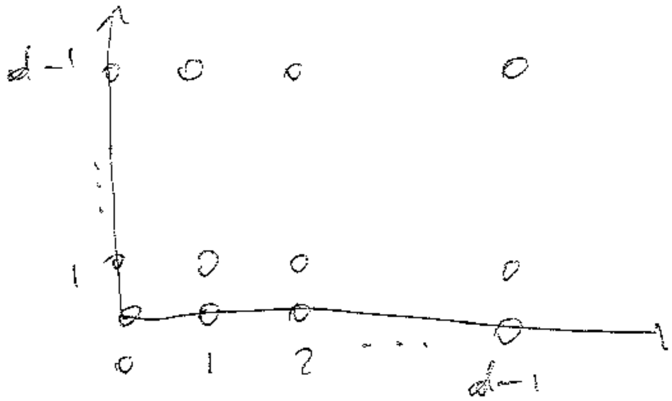
$$\forall_\alpha \quad \sum_{k=0}^{d-1} p_k^{(\alpha)} = 1 \quad \longrightarrow \quad (d-1) \text{ – niezależnych param.}$$

$$\text{Razem} = \underbrace{(d+1)}_{\text{ilość baz}} \cdot \underbrace{(d-1)}_{\substack{\text{ilość} \\ \text{niezależnych} \\ \text{parametrów}}} \equiv d^2 - 1$$

$d^2 - 1 \equiv$ # liczba niezależnych parametrów $\hat{\rho}$!

TOMOGRAFIA ?

Dyskretne przestrzenie fazowe



zbiór d^2 punktów

$e_0, \dots, e_{d-1}$ – wybrana baza w $\mathbb{C}^d$

$$\hat{X} e_\alpha = e_{\alpha+1} \qquad \text{mod } d$$

$$\hat{Z} e_\alpha = \omega^\alpha e_\alpha$$

$$\omega = e^{2\pi i/d}$$

<u>Przykład</u> $\quad d = 2$

$$\hat{X} e_0 = e_1$$
$$\hat{X} e_1 = e_0$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x$$

$$\omega = e^{2\pi i/2} = e^{\pi i} = -1$$

$$\hat{Z} e_0 = \omega^0 e_0 = e_0$$
$$\hat{Z} e_1 = \omega^1 e_1 = -e_1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$$

$\alpha, \beta \in 0, 1, \dots, d-1$

$$\boxed{\hat{D}(\alpha,\beta) = \omega^{-\alpha\beta/2} \, \hat{Z}^{\alpha} \, \hat{X}^{\beta}}$$

analog operatora przesunięcia.

$\boxed{d > 2}$

$$\hat{A}(0,0) = \sum_{\alpha=0}^{d-1} |\alpha \rangle \langle -\alpha|$$

– operator parzystości

$$\hat{A}(0,0)|\alpha\rangle = |-\alpha\rangle \qquad \text{mod } d$$

$$\hat{A}(\alpha,\beta) = \hat{D}(\alpha,\beta)\,\hat{A}(0,0)\,\hat{D}(\alpha,\beta)^{\dagger}$$

Własności:

1) $\hat{A}(\alpha,\beta)^{\dagger} = \hat{A}(\alpha,\beta)$

2) $\mathrm{Tr}\,\hat{A}(\alpha,\beta) = 1$

3) $\mathrm{Tr}\left[\hat{A}(\alpha,\beta)\,\hat{A}(\alpha',\beta')\right] = d\,\delta_{\alpha\alpha'}\,\delta_{\beta\beta'}$

$$\hat{\varrho} \longrightarrow \quad W_{\alpha\beta} = \frac{1}{d}\,\mathrm{Tr}\left[\hat{\varrho}\,\hat{A}(\alpha,\beta)\right]$$

$$\hat{\varrho} = \sum_{\alpha,\beta=0}^{d-1} W_{\alpha\beta}\,\hat{A}(\alpha,\beta)$$

$W_{\alpha\beta}$ - dyskretna f. Wignera.

Własności:

1) $W_{\alpha\beta} \in \mathbb{R}$ $\qquad \left(W(x,p) \in \mathbb{R} \right)$

2) $\sum_{\alpha,\beta} W_{\alpha\beta} = 1$ $\qquad \left(\int dx\,dp\, W(x,p) = 1 \right)$

3) $W_{\alpha\beta} \ngeqslant 0$ $\qquad \left(W(x,p) \ngeqslant 0 \right)$

4) $\hat{\rho}_1 \longrightarrow W^{(1)}$

$\hat{\rho}_2 \longrightarrow W^{(2)}$

$$\mathrm{Tr}\left(\hat{\rho}_1 \cdot \hat{\rho}_2\right) = d \sum_{\alpha,\beta} W^{(1)}_{\alpha\beta}\, W^{(2)}_{\alpha\beta}$$

$$\left(\int W_1 W_2\, dx\,dp = \frac{1}{h} \mathrm{Tr}\left[\hat{\rho}_1 \hat{\rho}_2\right] \right)$$

Linie w przestrzeni fazowej $d \times d$

- Linia – zbiór d różnych punktów
- $L_1 \parallel L_2$ jeśli nie posiadają punktów wspólnych.

$$d = 2$$

$$d = 3$$

$W_{\alpha\beta}$ – <u>nie jest</u> rozkładem prawd. na Γ

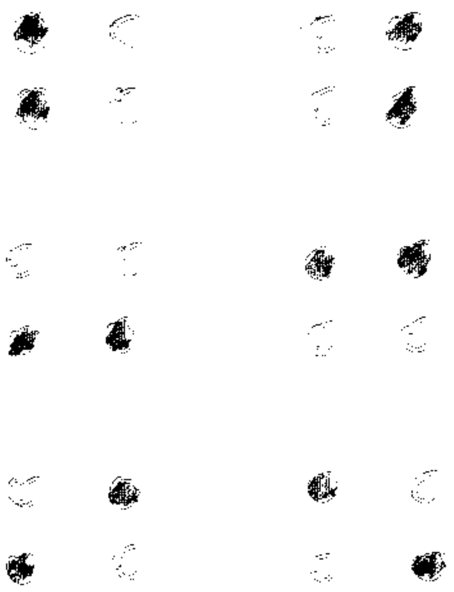
ale każda rodzina linii równoległych – TAK
$\{L_1, \dots, L_n\}$ – rodzina linii !!

$$\sum_{\alpha, \beta \in L_k} W_{\alpha\beta} = p_k$$

1) $p_k \geqslant 0$

2) $\sum p_k = 1$

[illegible]



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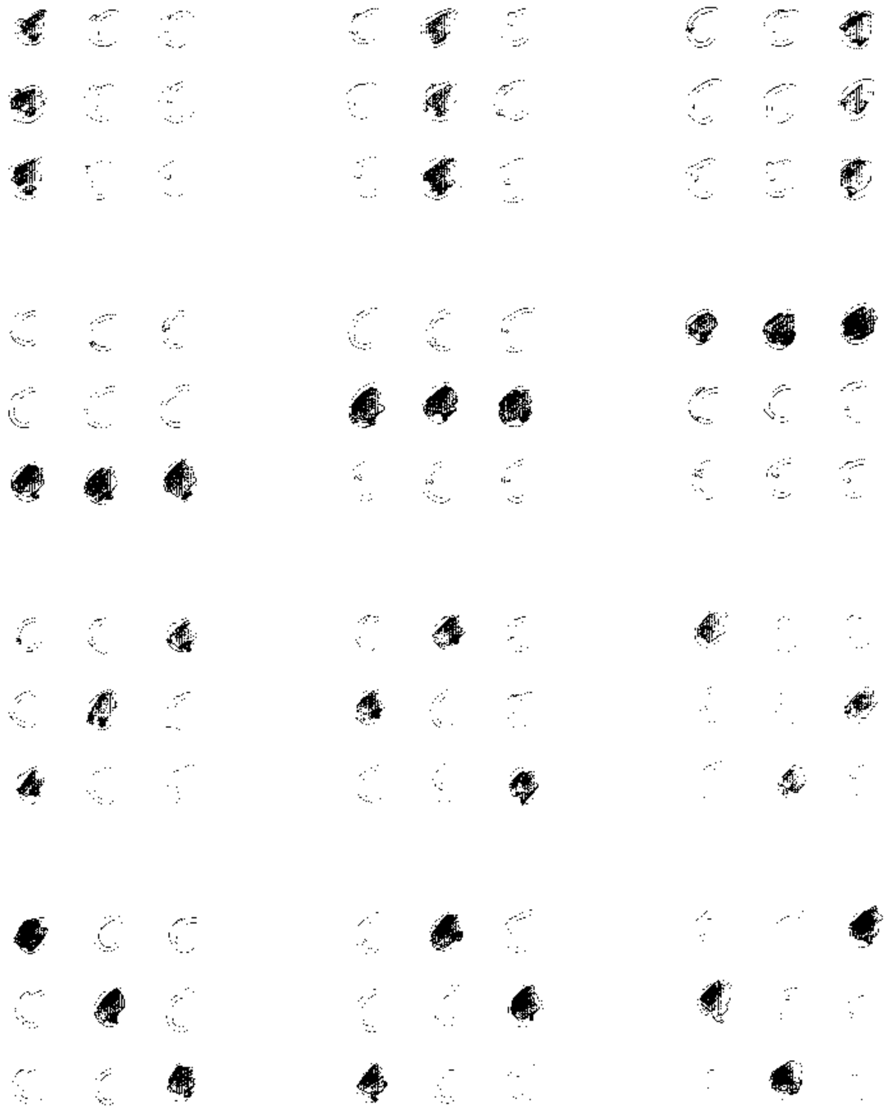

SIC – POVMs

Symmetric Informationally Complete.

$$\left.\begin{aligned} \Pi_i &\geq 0 \\ \sum_i \Pi_i &= \mathbb{I} \end{aligned}\right\}$$

Inf. complete

$$p_i = \mathrm{Tr}(\hat{\rho}\Pi_i) \longrightarrow \hat{\rho} = \ldots$$

Symmetric

$$\Pi_i = \frac{1}{d}|\psi_i\rangle\langle\psi_i| \qquad i = 1, \ldots, d^2$$

oraz $\quad |\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{d+1} \quad ; \quad i \neq j$.

FAKTY

1°. Analitycznie znaleziono dla
$d = 2 - 13,\ 15,\ 19$

2° Numerycznie $\quad d \leq 45$

3° Czy istnieją dla dowolnego d?